



**HAL**  
open science

# ON THE FRACTIONAL LANE-EMDEN EQUATION

Juan Davila, Louis Dupaigne, Juncheng Wei

► **To cite this version:**

Juan Davila, Louis Dupaigne, Juncheng Wei. ON THE FRACTIONAL LANE-EMDEN EQUATION. Transactions of the American Mathematical Society, 2016. hal-01283992

**HAL Id: hal-01283992**

**<https://sorbonne-paris-nord.hal.science/hal-01283992>**

Submitted on 7 Mar 2016

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON THE FRACTIONAL LANE-EMDEN EQUATION

JUAN DÁVILA, LOUIS DUPAIGNE, AND JUNCHENG WEI

ABSTRACT. We classify solutions of finite Morse index of the fractional Lane-Emden equation

$$(-\Delta)^s u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n.$$

## 1. INTRODUCTION

Fix an integer  $n \geq 1$  and let  $p_S(n)$  denote the classical Sobolev exponent:

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leq 2 \\ \frac{n+2}{n-2} & \text{if } n \geq 3 \end{cases}$$

A celebrated result of Gidas and Spruck [20] asserts that there is no positive solution to the Lane-Emden equation

$$(1.1) \quad -\Delta u = |u|^{p-1}u \quad \text{in } \mathbb{R}^n,$$

whenever  $p \in (1, p_S(n))$ . For  $p = p_S(n)$ , the same equation is known to have (up to translation and rescaling) a unique positive solution, which is radial and explicit (see Caffarelli-Gidas-Spruck [4]). Let now  $p_c(n) > p_S(n)$  denote the Joseph-Lundgren exponent:

$$p_c(n) = \begin{cases} +\infty & \text{if } n \leq 10 \\ \frac{(n-2)^2 - 4n + 8\sqrt{n-1}}{(n-2)(n-10)} & \text{if } n \geq 11 \end{cases}$$

This exponent can be characterized as follows: for  $p \geq p_S(n)$ , the explicit singular solution  $u_s(x) = A|x|^{-\frac{2}{p-1}}$  is unstable if and only if  $p < p_c(n)$ . It was proved by Farina [18] that (1.1) has no nontrivial finite Morse index solution whenever  $1 < p < p_c(n)$ ,  $p \neq p_S(n)$ .

Through blow-up analysis, such Liouville-type theorems imply interior regularity for solutions of a large class of semilinear elliptic equations: they are known to be equivalent to universal estimates for solutions of

$$(1.2) \quad -Lu = f(x, u, \nabla u) \quad \text{in } \Omega,$$

where  $L$  is a uniformly elliptic operator with smooth coefficients, the nonlinearity  $f$  scales like  $|u|^{p-1}u$  for large values of  $u$ , and  $\Omega$  is an open set of  $\mathbb{R}^n$ . For precise statements, see the work of Polacik, Quittner and Souplet [26] in the subcritical setting, as well as its adaptation to the supercritical case by Farina and two of the authors [11].

In the present work, we are interested in understanding whether similar results hold for equations involving a nonlocal diffusion operator, the simplest of which

is perhaps the fractional laplacian. Given  $s \in (0, 1)$ , the fractional version of the Lane-Emden equation<sup>1</sup> reads

$$(1.3) \quad (-\Delta)^s u = |u|^{p-1} u \quad \text{in } \mathbb{R}^n.$$

Here we have assumed that  $u \in C^{2\sigma}(\mathbb{R}^n)$ ,  $\sigma > s$  and

$$(1.4) \quad \int_{\mathbb{R}^n} \frac{|u(y)|}{(1+|y|)^{n+2s}} dy < +\infty,$$

so that the fractional laplacian of  $u$

$$(-\Delta)^s u(x) := \mathcal{A}_{n,s} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x-y|^{n+2s}} dy$$

is well-defined (in the principal-value sense) at every point  $x \in \mathbb{R}^n$ . The normalizing constant  $\mathcal{A}_{n,s} = \frac{2^{2s-1} \Gamma(\frac{n+2s}{2})}{\pi^{n/2} |\Gamma(-s)|}$  is of the order of  $s(1-s)$  as  $s$  converges to either 0 or 1.

The aforementioned classification results of Gidas-Spruck and Caffarelli-Gidas-Spruck have been generalized to the fractional setting (see Y. Li [24] and Chen-Li-Ou [8]). The corresponding fractional Sobolev exponent is given by

$$p_S(n) = \begin{cases} +\infty & \text{if } n \leq 2s \\ \frac{n+2s}{n-2s} & \text{if } n > 2s \end{cases}$$

Our main result is the following Liouville-type theorem for the fractional Lane-Emden equation.

**Theorem 1.1.** *Assume that  $n \geq 1$  and  $0 < s < \sigma < 1$ . Let  $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1+|y|)^{n+2s} dy)$  be a solution to (1.3) which is stable outside a compact set i.e. there exists  $R_0 \geq 0$  such that for every  $\varphi \in C_c^1(\mathbb{R}^n \setminus \overline{B_{R_0}})$ ,*

$$(1.5) \quad p \int_{\mathbb{R}^n} |u|^{p-1} \varphi^2 dx \leq \|\varphi\|_{\dot{H}^s(\mathbb{R}^n)}^2.$$

- If  $1 < p < p_S(n)$  or if  $p_S(n) < p$  and

$$(1.6) \quad p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

then  $u \equiv 0$ ;

- If  $p = p_S(n)$ , then  $u$  has finite energy i.e.

$$\|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 = \int_{\mathbb{R}^n} |u|^{p+1} < +\infty.$$

If in addition  $u$  is stable, then in fact  $u \equiv 0$ .

*Remark 1.* For  $p > p_S(n)$ , the function

$$(1.7) \quad u_s(x) = A|x|^{-\frac{2s}{p-1}}$$

<sup>1</sup>Unlike local diffusion operators, local elliptic regularity for equations of the form (1.2) where this time  $L$  is the generator of a general Markov diffusion, cannot be captured from the sole understanding of the fractional Lane-Emden equation. For example, further investigations will be needed for operators of Lévy symbol  $\psi(\xi) = \int_{S^{n-1}} |\omega \cdot \xi|^{2s} \mu(d\omega)$ , where  $\mu$  is any finite symmetric measure on the sphere  $S^{n-1}$ .

where

$$A^{p-1} = \lambda \left( \frac{n-2s}{2} - \frac{2s}{p-1} \right)$$

and where

$$(1.8) \quad \lambda(\alpha) = 2^{2s} \frac{\Gamma(\frac{n+2s+2\alpha}{4})\Gamma(\frac{n+2s-2\alpha}{4})}{\Gamma(\frac{n-2s-2\alpha}{4})\Gamma(\frac{n-2s+2\alpha}{4})}$$

is a singular solution to (1.3) (see the work by Montenegro and two of the authors [12] for the case  $s = 1/2$ , and the work by Fall [16, Lemma 3.1] for the general case). In virtue of the following Hardy inequality (due to Herbst [22])

$$\Lambda_{n,s} \int_{\mathbb{R}^n} \frac{\phi^2}{|x|^{2s}} dx \leq \|\phi\|_{H^s(\mathbb{R}^n)}^2$$

with optimal constant given by

$$\Lambda_{n,s} = 2^{2s} \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

$u_s$  is unstable if only if (1.6) holds. Let us also mention that regular radial solutions in the case  $s = 1/2$  were constructed by Chipot, Chlebik and Shafrir [9]. Recently, Harada [21] proved that for  $s = 1/2$ , condition (1.6) is the dividing line for the asymptotic behavior of radial solutions to (1.3), extending thereby the classical results of Joseph and Lundgren [23] to the fractional Lane-Emden equation in the case  $s = 1/2$ . A similar technique as in [9] allows us to show that the condition (1.6) is optimal. Indeed we have:

**Theorem 1.2.** *Assume  $p > p_S(n)$  and that (1.6) fails. Then there are radial smooth solutions  $u > 0$  with  $u(r) \rightarrow 0$  as  $r \rightarrow \infty$  of (1.3) that are stable.*

It is by now standard knowledge that the fractional laplacian can be seen as a Dirichlet-to-Neumann operator for a degenerate but *local* diffusion operator in the higher-dimensional half-space  $\mathbb{R}_+^{n+1}$ :

**Theorem 1.3** ([5, 25, 28]). *Take  $s \in (0, 1)$ ,  $\sigma > s$  and  $u \in C^{2\sigma}(\mathbb{R}^n) \cap L^1(\mathbb{R}^n, (1 + |y|)^{n+2s} dy)$ . For  $X = (x, t) \in \mathbb{R}_+^{n+1}$ , let*

$$\bar{u}(X) = \int_{\mathbb{R}^n} P(X, y)u(y) dy,$$

where

$$P(X, y) = p_{n,s} t^{2s} |X - y|^{-(n+2s)}$$

and  $p_{n,s}$  is chosen so that  $\int_{\mathbb{R}^n} P(X, y) dy = 1$ . Then,  $\bar{u} \in C^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$ ,  $t^{1-2s} \partial_t \bar{u} \in C(\overline{\mathbb{R}_+^{n+1}})$  and

$$\begin{cases} \nabla \cdot (t^{1-2s} \nabla \bar{u}) = 0 & \text{in } \mathbb{R}_+^{n+1}, \\ \bar{u} = u & \text{on } \partial \mathbb{R}_+^{n+1}, \\ -\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{u} = \kappa_s (-\Delta)^s u & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

where

$$(1.9) \quad \kappa_s = \frac{\Gamma(1-s)}{2^{2s-1} \Gamma(s)}.$$

Applying Theorem 1.3 to a solution of the fractional Lane-Emden equation, we end up with the equation

$$(1.10) \quad \begin{cases} -\nabla \cdot (t^{1-2s} \nabla \bar{u}) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{u} = \kappa_s |\bar{u}|^{p-1} \bar{u} & \text{on } \partial \mathbb{R}_+^{n+1} \end{cases}$$

An essential ingredient in the proof of Theorem 1.1 is the following monotonicity formula

**Theorem 1.4.** *Take a solution to (1.10)  $\bar{u} \in C^2(\mathbb{R}_+^{n+1}) \cap C(\overline{\mathbb{R}_+^{n+1}})$  such that  $t^{1-2s} \partial_t \bar{u} \in C(\overline{\mathbb{R}_+^{n+1}})$ . For  $x_0 \in \partial \mathbb{R}_+^{n+1}$ ,  $\lambda > 0$ , let*

$$E(\bar{u}, x_0; \lambda) = \lambda^{2s \frac{p+1}{p-1} - n} \left( \frac{1}{2} \int_{\mathbb{R}_+^{n+1} \cap B^{n+1}(x_0, \lambda)} t^{1-2s} |\nabla \bar{u}|^2 dx dt - \frac{\kappa_s}{p+1} \int_{\partial \mathbb{R}_+^{n+1} \cap B^{n+1}(x_0, \lambda)} |\bar{u}|^{p+1} dx \right) + \lambda^{2s \frac{p+1}{p-1} - n - 1} \frac{s}{p+1} \int_{\partial B^{n+1}(x_0, \lambda) \cap \mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2 d\sigma.$$

Then,  $E$  is a nondecreasing function of  $\lambda$ . Furthermore,

$$\frac{dE}{d\lambda} = \lambda^{2s \frac{p+1}{p-1} - n + 1} \int_{\partial B^{n+1}(x_0, \lambda) \cap \mathbb{R}_+^{n+1}} t^{1-2s} \left( \frac{\partial \bar{u}}{\partial r} + \frac{2s}{p-1} \frac{\bar{u}}{r} \right)^2 d\sigma$$

*Remark 2.* In the above,  $B^{n+1}(x_0, \lambda)$  denotes the euclidean ball in  $\mathbb{R}^{n+1}$  centered at  $x_0$  of radius  $\lambda$ ,  $\sigma$  the  $n$ -dimensional Hausdorff measure restricted to  $\partial B^{n+1}(x_0, \lambda)$ ,  $r = |X|$  the euclidean norm of a point  $X = (x, t) \in \mathbb{R}_+^{n+1}$ , and  $\partial_r = \nabla \cdot \frac{X}{r}$  the corresponding radial derivative.

An analogous monotonicity formula has been derived by Fall and Felli [17] to obtain unique continuation results for fractional equations. Previously, Caffarelli and Silvestre derived an Almgren quotient formula for the fractional laplacian in [5] and Caffarelli, Roquejoffre and Savin [6] obtained a related monotonicity formula to study regularity of nonlocal minimal surfaces. Another monotonicity formula for fractional problems was obtained by Cabré and Sire [3] and used by Frank, Lenzmann and Silvestre [19].

The proof of Theorem 1.1 follows an approach used in our earlier work with Kelei Wang [13] (see also [29]). First we derive suitable energy estimate (Section 2) and handle the critical and subcritical cases (Section 3). In the supercritical case, we make crucial use of a monotonicity formula Theorem 1.4, proved in Section 4. Thanks to it, using a blown-down analysis (first three steps of Section 6), we prove that the blow-down limit of a given solution is homogeneous. We then exclude the existence of stable homogeneous singular solutions in the optimal range of  $p$  (Section 5). Steps 5 and 6 of Section 6) are eventually devoted to prove that the solution itself is trivial. Finally we prove Theorem 1.2 in Section 7.

## 2. ENERGY ESTIMATES

**Lemma 2.1.** *Let  $u$  be a solution to (1.3). Assume that  $u$  is stable outside some ball  $B_{R_0}^n \subset \mathbb{R}^n$ . Let  $\eta \in C_c^\infty(\mathbb{R}^n \setminus \overline{B_{R_0}^n})$  and for  $x \in \mathbb{R}^n$ , define*

$$(2.1) \quad \rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x - y|^{n+2s}} dy$$

Then,

$$\int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx + \frac{1}{p} \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 \leq \frac{\mathcal{A}_{n,s}}{p-1} \int_{\mathbb{R}^n} u^2 \rho dx.$$

*Proof.* Multiply (1.3) by  $u\eta^2$ . Then,

$$\begin{aligned} \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx &= \int_{\mathbb{R}^n} (-\Delta)^s u u \eta^2 dx \\ &= \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x) - u(y))(u(x)\eta(x)^2 - u(y)\eta(y)^2)}{|x-y|^{n+2s}} dx dy \\ &= \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u^2(x)\eta^2(x) - u(x)u(y)(\eta^2(x) + \eta^2(y)) + u^2(y)\eta^2(y)}{|x-y|^{n+2s}} dx dy \\ &= \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(u(x)\eta(x) - u(y)\eta(y))^2 - (\eta(x) - \eta(y))^2 u(x)u(y)}{|x-y|^{n+2s}} dx dy \\ &= \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 - \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2 u(x)u(y)}{|x-y|^{n+2s}} dx dy \end{aligned}$$

Using the inequality  $2ab \leq a^2 + b^2$ , we deduce that

$$(2.2) \quad \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 - \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx \leq \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} u^2 \rho dx$$

Since  $u$  is stable, we deduce that

$$(p-1) \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx \leq \frac{\mathcal{A}_{n,s}}{2} \int_{\mathbb{R}^n} u^2 \rho dx$$

Going back to (2.2), it follows that

$$\frac{1}{p} \|u\eta\|_{\dot{H}^s(\mathbb{R}^n)}^2 + \int_{\mathbb{R}^n} |u|^{p+1} \eta^2 dx \leq \frac{\mathcal{A}_{n,s}}{p-1} \int_{\mathbb{R}^n} u^2 \rho dx$$

□

**Lemma 2.2.** For  $m > n/2$  and  $x \in \mathbb{R}^n$ , let

$$(2.3) \quad \eta(x) = (1 + |x|^2)^{-m/2} \quad \text{and} \quad \rho(x) = \int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy$$

Then, there exists a constant  $C = C(n, s, m) > 0$  such that

$$(2.4) \quad C^{-1} (1 + |x|^2)^{-\frac{n}{2}-s} \leq \rho(x) \leq C (1 + |x|^2)^{-\frac{n}{2}-s}.$$

*Proof.* Let us prove the upper bound first. Since  $\rho$  is a continuous function, we may always assume that  $|x| \geq 1$ . Split the integral

$$\int_{\mathbb{R}^n} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy$$

in the regions  $|x-y| < |x|/2$ ,  $|x|/2 \leq |x-y| \leq 2|x|$ , and  $|x-y| > 2|x|$ . When  $|x-y| \leq |x|/2$ ,

$$|\eta(x) - \eta(y)| \leq C(1 + |x|^2)^{-\frac{m+1}{2}} |x-y|.$$

So,

$$\begin{aligned} \int_{|x-y| \leq |x|/2} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy &\leq C(1 + |x|^2)^{-m-1} \int_{|x-y| \leq |x|/2} |x-y|^{2-n-2s} dy \\ &\leq C(1 + |x|^2)^{-m-s} \leq C(1 + |x|^2)^{-\frac{n}{2}-s}. \end{aligned}$$

When  $|x|/2 \leq |x-y| \leq 2|x|$ ,

$$\begin{aligned} \int_{|x|/2 \leq |x-y| \leq 2|x|} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy &\leq C|x|^{-n-2s} \int_{|y| \leq 2|x|} (\eta(x)^2 + \eta(y)^2) dy \\ &\leq C|x|^{-n-2s} (|x|^{-2m+n} + 1) \leq C(1 + |x|^2)^{-\frac{n}{2}-s}, \end{aligned}$$

where we used the assumption  $m > \frac{n}{2}$ . When  $|x-y| > 2|x|$ , then  $|y| \geq |x|$  and  $\eta(y) \leq C(1 + |x|^2)^{-m/2}$ . Then,

$$\begin{aligned} \int_{|x-y| > 2|x|} \frac{(\eta(x) - \eta(y))^2}{|x-y|^{n+2s}} dy &\leq C(1 + |x|^2)^{-m} \int_{|x-y| > 2|x|} \frac{1}{|x-y|^{n+2s}} dy \\ &\leq C(1 + |x|^2)^{-m-s} \leq C(1 + |x|^2)^{-\frac{n}{2}-s}. \end{aligned}$$

Let us turn to the lower bound. Again, we may always assume that  $|x| \geq 1$ . Then,

$$\rho(x) \geq \int_{|y| \leq 1/2} \frac{(\eta(y) - \eta(x))^2}{|x-y|^{n+2s}} dy \geq \left(\frac{|x|}{2}\right)^{-(n+2s)} \int_{|y| \leq 1/2} (\eta(y) - 2^{-m/2})^2 dy$$

and the estimate follows.  $\square$

**Corollary 2.3.** *Let  $m > n/2$ ,  $\eta$  given by (2.3),  $R \geq R_0 \geq 1$ ,  $\psi \in C^\infty(\mathbb{R}^n)$  be such that  $0 \leq \psi \leq 1$ ,  $\psi \equiv 0$  on  $B_1^n$  and  $\psi \equiv 1$  on  $\mathbb{R}^n \setminus B_2^n$ . Let*

$$(2.5) \quad \eta_R(x) = \eta\left(\frac{x}{R}\right) \psi\left(\frac{x}{R_0}\right) \quad \text{and} \quad \rho_R(x) = \int_{\mathbb{R}^n} \frac{(\eta_R(x) - \eta_R(y))^2}{|x-y|^{n+2s}} dy$$

*There exists a constant  $C = C(n, s, m, R_0) > 0$  such that for all  $|x| \geq 3R_0$*

$$\rho_R(x) \leq C\eta\left(\frac{x}{R}\right)^2 |x|^{-(n+2s)} + R^{-2s} \rho\left(\frac{x}{R}\right)$$

*Proof.* Fix  $x$  such that  $|x| \geq 3R_0$ . Using the definition of  $\eta_R$  and Young's inequality, we have

$$\begin{aligned} \frac{1}{2}\rho_R(x) &\leq \eta\left(\frac{x}{R}\right)^2 \int_{\mathbb{R}^n} \frac{\left(\psi\left(\frac{x}{R_0}\right) - \psi\left(\frac{y}{R_0}\right)\right)^2}{|x-y|^{n+2s}} dy + \int_{\mathbb{R}^n} \psi\left(\frac{y}{R_0}\right)^2 \frac{(\eta\left(\frac{x}{R}\right) - \eta\left(\frac{y}{R}\right))^2}{|x-y|^{n+2s}} dy \\ &\leq \eta\left(\frac{x}{R}\right)^2 \int_{B_{2R_0}^n} \frac{1}{|x-y|^{n+2s}} dy + \int_{\mathbb{R}^n} \frac{(\eta\left(\frac{x}{R}\right) - \eta\left(\frac{y}{R}\right))^2}{|x-y|^{n+2s}} dy \\ &\leq C\eta\left(\frac{x}{R}\right)^2 |x|^{-(n+2s)} + R^{-2s} \rho\left(\frac{x}{R}\right) \end{aligned}$$

and the result follows.  $\square$

**Lemma 2.4.** *Let  $u$  be a solution of (1.3) which is stable outside a ball  $B_{R_0}^n$ . Take  $\rho_R$  as in Corollary 2.3 with  $m \in (\frac{n}{2}, \frac{n}{2} + \frac{s(p+1)}{2})$ . Then, there exists a constant  $C = C(n, s, m, p, R_0) > 0$  such that for all  $R \geq 3R_0$ ,*

$$\int_{\mathbb{R}^n} u^2 \rho_R dx \leq C \left( \int_{B_{3R_0}^n} u^2 \rho_R dx + R^{n-2s} \frac{p+1}{p-1} \right).$$

*Proof.* By Corollary 2.3, if  $R \geq |x| \geq 3R_0$ , then

$$\rho_R(x) \leq C(|x|^{-n-2s} + R^{-2s})$$

and so

$$\int_{B_R^n \setminus B_{3R_0}^n} \rho_R(x)^{\frac{p+1}{p-1}} \eta_R(x)^{-\frac{4}{p-1}} dx \leq CR^{n-2s} \frac{p+1}{p-1}.$$

Similarly, if  $|x| \geq R \geq 3R_0$ , then

$$\rho_R(x) \leq CR^{-2s} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{n}{2}-s}$$

and so

$$\rho_R(x)^{\frac{p+1}{p-1}} \eta_R(x)^{-\frac{4}{p-1}} \leq CR^{-2s} \frac{p+1}{p-1} \left(1 + \frac{|x|^2}{R^2}\right)^{-\frac{n+2s}{2} \frac{p+1}{p-1} + \frac{2m}{p-1}}$$

Since  $m \in (\frac{n}{2}, \frac{n}{2} + s \frac{p+1}{2})$ , we have  $\frac{2m}{p-1} - \frac{n+2s}{2} \frac{p+1}{p-1} < -\frac{n}{2}$  and so

$$\int_{\mathbb{R}^n \setminus B_{3R_0}^n} \rho_R(x)^{\frac{p+1}{p-1}} \eta_R(x)^{-\frac{4}{p-1}} dx \leq CR^{n-2s} \frac{p+1}{p-1}.$$

Now,

$$\begin{aligned} \int_{\mathbb{R}^n} u^2 \rho_R dx &= \int_{B_{3R_0}^n} u^2 \rho_R dx + \int_{\mathbb{R}^n \setminus B_{3R_0}^n} u^2 \rho_R \eta_R^{-\frac{4}{p+1}} \eta_R^{\frac{4}{p+1}} dx \\ &\leq \int_{B_{3R_0}^n} u^2 \rho_R dx + \left( \int_{\mathbb{R}^n} |u|^{p+1} \eta_R^2 dx \right)^{\frac{2}{p+1}} \left( \int_{\mathbb{R}^n} \rho_R^{\frac{p+1}{p-1}} \eta_R^{-\frac{4}{p-1}} dx \right)^{\frac{p-1}{p+1}} \\ &\leq \int_{B_{3R_0}^n} u^2 \rho_R dx + CR^{(n-2s) \frac{p+1}{p-1} \frac{p-1}{p+1}} \left( \int_{\mathbb{R}^n} |u|^{p+1} \eta_R^2 dx \right)^{\frac{2}{p+1}} \end{aligned}$$

By a standard approximation argument, Lemma 2.1 remains valid with  $\eta = \eta_R$  and  $\rho = \rho_R$  and so the result follows.  $\square$

**Lemma 2.5.** *Assume that  $p \neq \frac{n+2s}{n-2s}$ . Let  $u$  be a solution to (1.3) which is stable outside a ball  $B_{R_0}^n$  and  $\bar{u}$  its extension, solving (1.10). Then, there exists a constant  $C = C(n, p, s, R_0, u) > 0$  such that*

$$\int_{B_R^{n+1}} t^{1-2s} \bar{u}^2 dx dt \leq CR^{n+2(1-s) - \frac{4s}{p-1}}$$

for any  $R > 3R_0$ .

*Proof.* According to Theorem 1.3,

$$\bar{u}(x, t) = p_{n,s} \int_{\mathbb{R}^n} u(z) \frac{t^{2s}}{(|x-z|^2 + t^2)^{\frac{n+2s}{2}}} dz$$

so that

$$\bar{u}(x, t)^2 \leq p_{n,s} \int_{\mathbb{R}^n} u(z)^2 \frac{t^{2s}}{(|x-z|^2 + t^2)^{\frac{n+2s}{2}}} dz.$$

So,

$$\begin{aligned} \int_{B_R^{n+1}} t^{1-2s} \bar{u}^2 dx dt &\leq p_{n,s} \int_{|x| \leq R, z \in \mathbb{R}^n} u(z)^2 \left( \int_0^R \frac{t}{(|x-z|^2 + t^2)^{\frac{n+2s}{2}}} dt \right) dz dx \\ &\leq C \int_{|x| \leq R, z \in \mathbb{R}^n} u^2(z) \left\{ (|x-z|^2)^{-\frac{n}{2}+1-s} - (|x-z|^2 + R^2)^{-\frac{n}{2}+1-s} \right\} dz dx \end{aligned}$$



Split this last integral according to  $|x - z| < 2R$  or  $|x - z| \geq 2R$ . Then,

$$\begin{aligned} & \int_{|x| \leq R, |x-z| < 2R} u^2(z) \left\{ (|x-z|^2)^{-\frac{n}{2}+1-s} - (|x-z|^2 + R^2)^{-\frac{n}{2}+1-s} \right\} dz dx \leq \\ & \int_{|x| \leq R, |x-z| < 2R} u^2(z) (|x-z|^2)^{-\frac{n}{2}+1-s} dz dx \leq CR^{2(1-s)} \int_{B_{3R}^{n+1}} u^2(z) dz \leq \\ & CR^{2(1-s)} \left( \int |u|^{p+1} \eta_R^2 \right)^{\frac{2}{p+1}} \left( \int_{B_{3R}^{n+1}} \eta_R^{-\frac{4}{p-1}} \right)^{\frac{p-1}{p+1}} \leq \\ & CR^{2(1-s)+n\frac{p-1}{p+1}} \left( \int u^2(z) \rho_R(z) dz \right)^{\frac{2}{p+1}} \leq CR^{n+2(1-s)-\frac{4s}{p-1}}, \end{aligned}$$

where we used Hölder's inequality, then Lemma 2.1 and then Lemma 2.4. For the region  $|x - z| \geq 2R$ , the mean-value inequality implies that

$$\begin{aligned} & \int_{|x| \leq R, |x-z| \geq 2R} u^2(z) \left\{ (|x-z|^2)^{-\frac{n}{2}+1-s} - (|x-z|^2 + R^2)^{-\frac{n}{2}+1-s} \right\} dz dx \leq \\ & CR^2 \int_{|x| \leq R, |x-z| \geq 2R} u^2(z) |x-z|^{-(n+2s)} dz dx \leq CR^{n+2} \int_{|z| \geq R} u^2(z) |z|^{-(n+2s)} dz \\ & \leq CR^2 \int_{|z| \geq R} u^2 \rho dz \leq CR^{n+2(1-s)-\frac{4s}{p-1}}, \end{aligned}$$

where we used again Corollary 2.3 in the penultimate inequality and Lemma 2.4 in the last one.  $\square$

**Lemma 2.6.** *Let  $u$  be a solution to (1.3) which is stable outside a ball  $B_{R_0}^n$  and  $\bar{u}$  its extension, solving (1.10). Then, there exists a constant  $C = C(n, p, s, u) > 0$  such that*

$$\int_{B_R^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} |\nabla \bar{u}|^2 dx dt + \int_{B_R^{n+1} \cap \partial \mathbb{R}_+^{n+1}} |u|^{p+1} dx \leq CR^{n-2s\frac{p+1}{p-1}}$$

*Proof.* The  $L^{p+1}$  estimate follows from Lemmata 2.1 and 2.4. Now take a cut-off function  $\eta \in C_c^1(\mathbb{R}_+^{n+1})$  such that  $\eta = 1$  on  $\mathbb{R}_+^{n+1} \cap (B_R^{n+1} \setminus B_{2R_0}^{n+1})$  and  $\eta = 0$  on  $B_{R_0}^{n+1} \cup (\mathbb{R}_+^{n+1} \setminus B_{2R}^{n+1})$ , and multiply equation (1.10) by  $\bar{u}\eta^2$ . Then,

$$\begin{aligned} (2.6) \quad \kappa_s \int_{\partial \mathbb{R}_+^{n+1}} |\bar{u}|^{p+1} \eta^2 dx &= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \{ \nabla \bar{u} \cdot \nabla (\bar{u}\eta^2) \} dx dt \\ &= \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \{ |\nabla(\bar{u}\eta)|^2 - \bar{u}^2 |\nabla \eta|^2 \} dx dt. \end{aligned}$$

Since  $u$  is stable outside  $B_{R_0}^{n+1}$ , so is  $\bar{u}$  and we deduce that

$$\frac{1}{p} \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla(\bar{u}\eta)|^2 dx dt \geq \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \{ |\nabla(\bar{u}\eta)|^2 - \bar{u}^2 |\nabla \eta|^2 \} dx dt.$$

In other words,

$$(2.7) \quad p' \int_{\mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2 |\nabla \eta|^2 dx dt \geq \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla(\bar{u}\eta)|^2 dx dt,$$

where  $\frac{1}{p'} + \frac{1}{p} = 1$ . We then apply Lemma 2.5.  $\square$

## 3. THE SUBCRITICAL CASE

In this section, we prove Theorem 1.1 for  $1 < p \leq p_S(n)$ .

*Proof.* Take a solution  $u$  which is stable outside some ball  $B_{R_0}^n$ . Apply Lemma 2.4 and let  $R \rightarrow +\infty$ . Since  $p \leq p_S(n)$ , we deduce that  $u \in \dot{H}^s(\mathbb{R}^n) \cap L^{p+1}(\mathbb{R}^n)$ . Multiplying the equation (1.3) by  $u$  and integrating, we deduce that

$$(3.1) \quad \int_{\mathbb{R}^n} |u|^{p+1} = \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2,$$

while multiplying by  $u^\lambda$  given for  $\lambda > 0$  and  $x \in \mathbb{R}^n$  by

$$u^\lambda(x) = u(\lambda x)$$

yields

$$\int_{\mathbb{R}^n} |u|^{p-1} u^\lambda = \int_{\mathbb{R}^n} (-\Delta)^{s/2} u (-\Delta)^{s/2} u^\lambda = \lambda^s \int_{\mathbb{R}^n} w w^\lambda,$$

where  $w = (-\Delta)^{s/2} u$ . Following Ros-Oton and Serra [27], we use the change of variable  $y = \sqrt{\lambda} x$  to deduce that

$$\lambda^s \int_{\mathbb{R}^n} w w^\lambda dx = \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} dy$$

Hence,

$$\begin{aligned} -\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} &= \int_{\mathbb{R}^n} x \cdot \nabla \frac{|u|^{p+1}}{p+1} = \int_{\mathbb{R}^n} (|u|^{p-1} u) x \cdot \nabla u = \\ &= \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} |u|^{p-1} u u^\lambda = \frac{d}{d\lambda} \Big|_{\lambda=1} \lambda^{\frac{2s-n}{2}} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} dy = \\ &= \frac{2s-n}{2} \int_{\mathbb{R}^n} w^2 + \frac{d}{d\lambda} \Big|_{\lambda=1} \int_{\mathbb{R}^n} w^{\sqrt{\lambda}} w^{1/\sqrt{\lambda}} dy = \frac{2s-n}{2} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2 \end{aligned}$$

In the last equality, we have used the fact that  $w \in C^1(\mathbb{R}^n)$ , as follows by elliptic regularity. We have just proved the following Pohozaev identity

$$\frac{n}{p+1} \int_{\mathbb{R}^n} |u|^{p+1} = \frac{n-2s}{2} \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2$$

For  $p < p_S(n)$ , the above identity together with (3.1) force  $u \equiv 0$ . For  $p = p_S(n)$ , we are left with proving that there is no stable nontrivial solution. Since  $u \in \dot{H}^s(\mathbb{R}^n)$ , we may apply the stability inequality (1.5) with test function  $\varphi = u$ , so that

$$p \int_{\mathbb{R}^n} |u|^{p+1} \leq \|u\|_{\dot{H}^s(\mathbb{R}^n)}^2.$$

This contradicts (3.1) unless  $u \equiv 0$ .  $\square$

In the following sections, we present several tools to study the supercritical case.

## 4. THE MONOTONICITY FORMULA

In this section, we prove Theorem 1.4.

*Proof.* Since the equation is invariant under translation, it suffices to consider the case where the center of the considered ball is the origin  $x_0 = 0$ . Let

$$(4.1) \quad E_1(\bar{u}; \lambda) = \lambda^{2s \frac{p+1}{p-1} - n} \left( \int_{\mathbb{R}_+^{n+1} \cap B_\lambda^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_\lambda^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}|^{p+1} dx \right)$$

For  $X \in \mathbb{R}_+^{n+1}$ , let also

$$(4.2) \quad U(X; \lambda) = \lambda^{\frac{2s}{p-1}} \bar{u}(\lambda X).$$

Then,  $U$  satisfies the three following properties:  $U$  solves (1.10),

$$(4.3) \quad E_1(\bar{u}; \lambda) = E_1(U; 1),$$

and, using subscripts to denote partial derivatives,

$$(4.4) \quad \lambda U_\lambda = \frac{2s}{p-1} U + r U_r.$$

Differentiating the right-hand side of (4.3), we find

$$\frac{dE_1}{d\lambda}(\bar{u}; \lambda) = \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1}} t^{1-2s} \nabla U \cdot \nabla U_\lambda dx dt - \kappa_s \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1}} |U|^{p-1} U_\lambda dx.$$

Integrating by parts and then using (4.4),

$$\begin{aligned} \frac{dE_1}{d\lambda}(\bar{u}; \lambda) &= \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U_r U_\lambda d\sigma \\ &= \lambda \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U_\lambda^2 d\sigma - \frac{2s}{p-1} \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U U_\lambda d\sigma \\ &= \lambda \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U_\lambda^2 d\sigma - \frac{s}{p-1} \left( \int_{\partial B_1^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} U^2 d\sigma \right)_\lambda \end{aligned}$$

Scaling back, the theorem follows.  $\square$

## 5. HOMOGENEOUS SOLUTIONS

**Theorem 5.1.** *Let  $\bar{u}$  be a stable homogeneous solution of (1.10). Assume that  $p > \frac{n+2s}{n-2s}$  and*

$$(5.1) \quad p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} > \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2}.$$

Then,  $\bar{u} \equiv 0$ .

*Proof.* Take standard polar coordinates in  $\mathbb{R}_+^{n+1}$ :  $X = (x, t) = r\theta$ , where  $r = |X|$  and  $\theta = \frac{X}{|X|}$ . Let  $\theta_1 = \frac{t}{|X|}$  denote the component of  $\theta$  in the  $t$  direction and  $S_+^n = \{X \in \mathbb{R}_+^{n+1} : r = 1, \theta_1 > 0\}$  denote the upper unit half-sphere.

**Step 1.** Let  $\bar{u}$  be a homogeneous solution of (1.10) i.e. assume that for some  $\psi \in C^2(S_+^n)$ ,

$$\bar{u}(X) = r^{-\frac{2s}{p-1}} \psi(\theta).$$

Then,

$$(5.2) \quad \int_{S_+^n} \theta_1^{1-2s} |\nabla \psi|^2 + \beta \int_{S_+^n} \theta_1^{1-2s} \psi^2 = \kappa_s \int_{\partial S_+^n} |\psi|^{p+1},$$

where  $\kappa_s$  is given by (1.9) and

$$\beta = \frac{2s}{p-1} \left( n - 2s - \frac{2s}{p-1} \right).$$

Indeed, since  $\bar{u}$  solves (1.10) and is homogeneous,  $\psi$  solves

$$(5.3) \quad \begin{cases} -\operatorname{div}(\theta_1^{1-2s} \nabla \psi) + \beta \theta_1^{1-2s} \psi = 0 & \text{on } S_+^n \\ -\lim_{\theta_1 \rightarrow 0} \theta_1^{1-2s} \partial_{\theta_1} \psi = \kappa_s |\psi|^{p-1} \psi & \text{on } \partial S_+^n, \end{cases}$$

Multiplying (5.3) by  $\psi$  and integrating, (5.2) follows.

**Step 2.** For all  $\varphi \in C^1(S_+^n)$ ,

$$(5.4) \quad \kappa_s p \int_{\partial S_+^n} |\psi|^{p-1} \varphi^2 \leq \int_{S_+^n} \theta_1^{1-2s} |\nabla \varphi|^2 + \left( \frac{n-2s}{2} \right)^2 \int_{S_+^n} \theta_1^{1-2s} \varphi^2$$

By definition,  $\bar{u}$  is stable if for all  $\phi \in C_c^1(\overline{\mathbb{R}_+^{n+1}})$ ,

$$(5.5) \quad \kappa_s p \int_{\partial \mathbb{R}_+^{n+1}} |\bar{u}|^{p-1} \phi^2 dx \leq \int_{\mathbb{R}_+^{n+1}} t^{1-2s} |\nabla \phi|^2 dx dt$$

Choose a standard cut-off function  $\eta_\epsilon \in C_c^1(\mathbb{R}_+^*)$  at the origin and at infinity i.e.  $\chi_{(\epsilon, 1/\epsilon)}(r) \leq \eta_\epsilon(r) \leq \chi_{(\epsilon/2, 2/\epsilon)}(r)$ . Let also  $\varphi \in C^1(S_+^n)$ , apply (5.5) with

$$\phi(X) = r^{-\frac{n-2s}{2}} \eta_\epsilon(r) \varphi(\theta) \quad \text{for } X \in \mathbb{R}_+^{n+1},$$

and let  $\epsilon \rightarrow 0$ . Inequality (5.4) follows.

**Step 3.** For  $\alpha \in (0, \frac{n-2s}{2})$ ,  $x \in \mathbb{R}^n \setminus \{0\}$ , let

$$v_\alpha(x) = |x|^{-\frac{n-2s}{2} + \alpha}$$

and  $\bar{v}_\alpha$  its extension, as defined in Theorem 1.3. Then,  $\bar{v}_\alpha$  is homogeneous i.e. there exists  $\phi_\alpha \in C^2(S_+^n)$  such that for  $X \in \mathbb{R}_+^{n+1} \setminus \{0\}$ ,

$$\bar{v}_\alpha(X) = r^{-\frac{n-2s}{2} + \alpha} \phi_\alpha(\theta).$$

In addition, for all  $\varphi \in C^1(S_+^n)$ ,

$$(5.6) \quad \begin{aligned} \int_{S_+^n} \theta_1^{1-2s} |\nabla \varphi|^2 + \left( \left( \frac{n-2s}{2} \right)^2 - \alpha^2 \right) \int_{S_+^n} \theta_1^{1-2s} \varphi^2 \\ = \kappa_s \lambda(\alpha) \int_{\partial S_+^n} \varphi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_\alpha^2 \left| \nabla \left( \frac{\varphi}{\phi_\alpha} \right) \right|^2 \end{aligned}$$

Indeed, according to Fall [16, Lemma 3.1],  $\bar{v}_\alpha$  is homogeneous. Using the calculus identity stated by Fall-Felli in [17, Lemma 2.1], we get

$$(5.7) \quad \begin{cases} -\operatorname{div}(\theta_1^{1-2s} \nabla \phi_\alpha) + \left( \left( \frac{n-2s}{2} \right)^2 - \alpha^2 \right) \theta_1^{1-2s} \phi_\alpha = 0 & \text{on } S_+^n \\ \phi_\alpha = 1 & \text{on } \partial S_+^n. \end{cases}$$

Multiply equation (5.7) by  $\varphi^2/\phi_\alpha$ , integrate by parts, apply the calculus identity

$$\nabla\phi_\alpha \cdot \nabla \frac{\varphi^2}{\phi_\alpha} = |\nabla\varphi|^2 - \left| \nabla \frac{\varphi}{\phi_\alpha} \right|^2 \phi_\alpha^2$$

and recall from Fall [16, Lemma 3.1] that

$$-\lim_{t \rightarrow 0} t^{1-2s} \partial_t \bar{v}_\alpha = \kappa_s \lambda(\alpha) |x|^{-\frac{n-2s}{2} + \alpha - 2s},$$

where  $\lambda(\alpha)$  is given by (1.8).

**Step 4.** For  $\alpha \in (0, \frac{n-2s}{2})$

$$(5.8) \quad \phi_0 \leq \phi_\alpha \quad \text{on } S_+^n.$$

Indeed, on  $S_+^n$ ,

$$\operatorname{div}(\theta_1^{1-2s} \nabla \phi_0) = \left( \frac{n-2s}{2} \right)^2 \theta_1^{1-2s} \phi_0 \geq \left( \left( \frac{n-2s}{2} \right)^2 - \alpha^2 \right) \theta_1^{1-2s} \phi_0$$

so  $\phi_0$  is a sub-solution of (5.7). By the maximum principle, the conclusion follows.

**Step 5.** End of proof. Fix  $\alpha \in (0, \frac{n-2s}{2})$  given by

$$\alpha = \frac{n-2s}{2} - \frac{2s}{p-1}$$

so that

$$\left( \frac{n-2s}{2} \right)^2 - \alpha^2 = \frac{2s}{p-1} \left( n-2s - \frac{2s}{p-1} \right) = \beta,$$

where  $\beta$  is the constant appearing in (5.3).

Use the stability inequality (5.4) with  $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$ :

$$(5.9) \quad \kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \int_{S_+^n} \theta_1^{1-2s} \left| \nabla \left( \frac{\psi\phi_0}{\phi_\alpha} \right) \right|^2 + \left( \frac{n-2s}{2} \right)^2 \int_{S_+^n} \theta_1^{1-2s} \left( \frac{\psi\phi_0}{\phi_\alpha} \right)^2.$$

Note that a particular case of the identity (5.6) is

$$(5.10) \quad \int_{S_+^n} \theta_1^{1-2s} |\nabla\varphi|^2 + \left( \frac{n-2s}{2} \right)^2 \int_{S_+^n} \theta_1^{1-2s} \varphi^2 = \kappa_s \Lambda_{n,s} \int_{\partial S_+^n} \varphi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_0^2 \left| \nabla \left( \frac{\varphi}{\phi_0} \right) \right|^2$$

Using (5.10) (with  $\varphi = \frac{\psi\phi_0}{\phi_\alpha}$ ), (5.9) becomes

$$\kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \kappa_s \Lambda_{n,s} \int_{\partial S_+^n} \psi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_0^2 \left| \nabla \left( \frac{\psi}{\phi_\alpha} \right) \right|^2.$$

By (5.8), we deduce that

$$\kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \kappa_s \Lambda_{n,s} \int_{\partial S_+^n} \psi^2 + \int_{S_+^n} \theta_1^{1-2s} \phi_\alpha^2 \left| \nabla \left( \frac{\psi}{\phi_\alpha} \right) \right|^2.$$

Using again the identity (5.6), we deduce that

$$\kappa_s p \int_{\partial S_+^n} |\psi|^{p+1} \leq \kappa_s (\Lambda_{n,s} - \lambda(\alpha)) \int_{\partial S_+^n} \psi^2 + \int_{S_+^n} \theta_1^{1-2s} |\nabla\psi|^2 + \beta \int_{S_+^n} \theta_1^{1-2s} \psi^2$$

Comparing with (5.2), it follows that

$$(5.11) \quad (p-1) \int_{\partial S_+^n} |\psi|^{p+1} \leq (\Lambda_{n,s} - \lambda(\alpha)) \int_{\partial S_+^n} \psi^2.$$

But from (5.2) and (5.6)

$$\int_{\partial S_+^n} |\psi|^{p+1} \geq \lambda(\alpha) \int_{\partial S_+^n} \psi^2$$

Combined with (5.11), we find that

$$\lambda(\alpha)p \leq \Lambda_{n,s}$$

unless  $\psi \equiv 0$ . □

## 6. BLOW-DOWN ANALYSIS

*Proof of Theorem 1.1.* Assume that  $p > p_S(n)$ . Take a solution  $u$  of (1.3) which is stable outside the ball of radius  $R_0$  and let  $\bar{u}$  be its extension solving (1.10).

**Step 1.**  $\lim_{\lambda \rightarrow +\infty} E(\bar{u}, 0; \lambda) < +\infty$ .

Since  $E$  is nondecreasing, it suffices to show that  $E(\bar{u}, 0; \lambda)$  is bounded. Write  $E = E_1 + E_2$ , where  $E_1$  is given by (4.1) and

$$E_2(\bar{u}; \lambda) = \lambda^{2s \frac{p+1}{p-1} - n - 1} \frac{s}{p+1} \int_{\partial B^{n+1}(0, \lambda) \cap \mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2 d\sigma$$

By Lemma 2.6,  $E_1$  is bounded. Since  $E$  is nondecreasing,

$$E(\bar{u}; \lambda) \leq \frac{1}{\lambda} \int_{\lambda}^{2\lambda} E(u; t) dt \leq C + \lambda^{2s \frac{p+1}{p-1} - n - 1} \int_{B_{2\lambda}^{n+1} \cap \mathbb{R}_+^{n+1}} t^{1-2s} \bar{u}^2.$$

Applying Lemma 2.5, we deduce that  $E$  is bounded.

**Step 2.** There exists a sequence  $\lambda_i \rightarrow +\infty$  such that  $(\bar{u}^{\lambda_i})$  converges weakly in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$  to a function  $\bar{u}^\infty$ .

This follows from the fact that  $(\bar{u}^{\lambda_i})$  is bounded in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$  by Lemma 2.6.

**Step 3.**  $\bar{u}^\infty$  is homogeneous

To see this, apply the scale invariance of  $E$ , its finiteness and the monotonicity formula: given  $R_2 > R_1 > 0$ ,

$$\begin{aligned} 0 &= \lim_{i \rightarrow +\infty} E(\bar{u}; \lambda_i R_2) - E(\bar{u}; \lambda_i R_1) \\ &= \lim_{i \rightarrow +\infty} E(\bar{u}^{\lambda_i}; R_2) - E(\bar{u}^{\lambda_i}; R_1) \\ &\geq \liminf_{i \rightarrow +\infty} \int_{(B_{R_2}^{n+1} \setminus B_{R_1}^{n+1}) \cap \mathbb{R}_+^{n+1}} t^{1-2s} r^{2-n+\frac{4s}{p-1}} \left( \frac{2s}{p-1} \frac{\bar{u}^{\lambda_i}}{r} + \frac{\partial \bar{u}^{\lambda_i}}{\partial r} \right)^2 dx dt \\ &\geq \int_{(B_{R_2}^{n+1} \setminus B_{R_1}^{n+1}) \cap \mathbb{R}_+^{n+1}} t^{1-2s} r^{2-n+\frac{4s}{p-1}} \left( \frac{2s}{p-1} \frac{\bar{u}^\infty}{r} + \frac{\partial \bar{u}^\infty}{\partial r} \right)^2 dx dt \end{aligned}$$

Note that in the last inequality we only used the weak convergence of  $(\bar{u}^{\lambda_i})$  to  $\bar{u}^\infty$  in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$ . So,

$$\frac{2s}{p-1} \frac{\bar{u}^\infty}{r} + \frac{\partial \bar{u}^\infty}{\partial r} = 0 \quad a.e. \text{ in } \mathbb{R}_+^{n+1}.$$

And so,  $u^\infty$  is homogeneous.

**Step 4.**  $\bar{u}^\infty \equiv 0$

Simply apply Theorem 5.1.

**Step 5.**  $(\bar{u}^{\lambda_i})$  converges strongly to zero in  $H^1(B_R^{n+1} \setminus B_\epsilon^{n+1}; t^{1-2s} dx dt)$  and  $(u^{\lambda_i})$  converges strongly to zero in  $L^{p+1}(B_R^{n+1} \setminus B_\epsilon^{n+1})$  for all  $R > \epsilon > 0$ . Indeed, by Steps 2 and 3,  $(\bar{u}^{\lambda_i})$  is bounded in  $H_{loc}^1(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$  and converges weakly to 0. It follows that  $(\bar{u}^{\lambda_i})$  converges strongly to 0 in  $L_{loc}^2(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$ . Indeed, by the standard Rellich-Kondrachov theorem and a diagonal argument, passing to a subsequence we obtain

$$\int_{\mathbb{R}_+^{n+1} \cap (B_R^{n+1} \setminus A)} t^{1-2s} |\bar{u}^{\lambda_i}|^2 dx dt \rightarrow 0,$$

as  $i \rightarrow \infty$ , for any  $B_R^{n+1} = B_R^{n+1}(0) \subset \mathbb{R}^{n+1}$  and  $A$  of the form  $A = \{(x, t) \in \mathbb{R}_+^{n+1} : 0 < t < r/2\}$ , where  $R, r > 0$ . By [15, Theorem 1.2],

$$\int_{\mathbb{R}_+^{n+1} \cap B_r^{n+1}(x)} t^{1-2s} |\bar{u}^{\lambda_i}|^2 dx dt \leq Cr^2 \int_{\mathbb{R}_+^{n+1} \cap B_r^{n+1}(x)} t^{1-2s} |\nabla \bar{u}^{\lambda_i}|^2 dx dt$$

for any  $x \in \partial \mathbb{R}_+^{n+1}$ ,  $|x| \leq R$ , with a uniform constant  $C$ . Covering  $B_R^{n+1} \cap A$  with half balls  $B_r^{n+1}(x) \cap \mathbb{R}_+^{n+1}$ ,  $x \in \partial \mathbb{R}_+^{n+1}$  with finite overlap, we see that

$$\int_{B_R^{n+1} \cap A} t^{1-2s} |\bar{u}^{\lambda_i}|^2 dx dt \leq Cr^2 \int_{B_R^{n+1} \cap A} t^{1-2s} |\nabla \bar{u}^{\lambda_i}|^2 dx dt \leq Cr^2,$$

and from this we conclude that  $(\bar{u}^{\lambda_i})$  converges strongly to 0 in  $L_{loc}^2(\mathbb{R}_+^{n+1}; t^{1-2s} dx dt)$ .

Now, using (2.7),  $(\bar{u}^{\lambda_i})$  converges strongly to 0 in  $H_{loc}^1(\mathbb{R}_+^{n+1} \setminus \{0\}; t^{1-2s} dx dt)$  and by (2.6), the convergence also holds in  $L_{loc}^{p+1}(\mathbb{R}^n \setminus \{0\})$ .

**Step 6.**  $\bar{u} \equiv 0$ .

Indeed,

$$\begin{aligned} E_1(\bar{u}; \lambda) &= E_1(\bar{u}^\lambda; 1) = \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \\ &= \int_{\mathbb{R}_+^{n+1} \cap B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx + \\ &\quad \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \\ &= \epsilon^{n-2s} \frac{p+1}{p-1} E_1(\bar{u}; \lambda \epsilon) + \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \\ &\leq C \epsilon^{n-2s} \frac{p+1}{p-1} + \int_{\mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} t^{1-2s} \frac{|\nabla \bar{u}^\lambda|^2}{2} dx dt - \int_{\partial \mathbb{R}_+^{n+1} \cap B_1^{n+1} \setminus B_\epsilon^{n+1}} \frac{\kappa_s}{p+1} |\bar{u}^\lambda|^{p+1} dx \end{aligned}$$

Letting  $\lambda \rightarrow +\infty$  and then  $\epsilon \rightarrow 0$ , we deduce that  $\lim_{\lambda \rightarrow +\infty} E_1(\bar{u}; \lambda) = 0$ . Using the monotonicity of  $E$ ,

$$E(\bar{u}; \lambda) \leq \frac{1}{\lambda} \int_\lambda^{2\lambda} E(t) dt \leq \sup_{[\lambda, 2\lambda]} E_1 + C \lambda^{-n-1+2s} \frac{p+1}{p-1} \int_{B_{2\lambda}^{n+1} \setminus B_\lambda^{n+1}} \bar{u}^2$$

and so  $\lim_{\lambda \rightarrow +\infty} E(\bar{u}; \lambda) = 0$ . Since  $u$  is smooth, we also have  $E(\bar{u}; 0) = 0$ . Since  $E$  is monotone,  $E \equiv 0$  and so  $\bar{u}$  must be homogeneous, a contradiction unless  $\bar{u} \equiv 0$ .  $\square$

## 7. CONSTRUCTION OF RADIAL ENTIRE STABLE SOLUTIONS

Let  $\bar{u}_s$  denote the extension of the singular solution  $u_s$  (1.7) to  $\mathbb{R}_+^{n+1}$  defined by

$$\bar{u}_s(X) = \int_{\mathbb{R}^n} P(X, y) u(y) dy.$$

Let  $B_1^{n+1}$  denote the unit ball in  $\mathbb{R}^{n+1}$  and for  $\lambda \geq 0$ , consider

$$(7.1) \quad \begin{cases} \operatorname{div}(t^{1-2s} \nabla u) = 0 & \text{in } B_1^{n+1} \cap \mathbb{R}_+^{n+1} \\ u = \lambda \bar{u}_s & \text{on } \partial B_1^{n+1} \cap \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} (t^{1-2s} u_t) = \kappa_s u^p & \text{on } B_1^{n+1} \cap \{t = 0\}. \end{cases}$$

Take  $\lambda \in (0, 1)$ . Since  $u_s$  is a positive supersolution of (7.1), there exists a minimal solution  $u = u_\lambda$ . By minimality, the family  $(u_\lambda)$  is nondecreasing and  $u_\lambda$  is axially symmetric, that is,  $u_\lambda(x, t) = u_\lambda(r, t)$  with  $r = |x| \in [0, 1]$ . In addition, for a fixed value  $\lambda \in (0, 1)$ ,  $u_\lambda$  is bounded, as can be proved by the truncation method of [1], see also [10] and radially decreasing by the moving plane method (see [7] for a similar setting). From now on let us assume that  $p_S(n) < p$  and

$$p \frac{\Gamma(\frac{n}{2} - \frac{s}{p-1}) \Gamma(s + \frac{s}{p-1})}{\Gamma(\frac{s}{p-1}) \Gamma(\frac{n-2s}{2} - \frac{s}{p-1})} \leq \frac{\Gamma(\frac{n+2s}{4})^2}{\Gamma(\frac{n-2s}{4})^2},$$

which means that the singular solution  $u_s$  is stable. Then,  $u_\lambda \uparrow u_s$  as  $\lambda \uparrow 1$ , using the classical convexity argument in [2] (see also Section 3.2.2 in [14]). Let  $\lambda_j \uparrow 1$  and

$$m_j = \|u_{\lambda_j}\|_{L^\infty} = u_{\lambda_j}(0), \quad R_j = m_j^{\frac{p-1}{2s}},$$

so that  $m_j, R_j \rightarrow \infty$  as  $j \rightarrow \infty$ . Set

$$v_j(x) = m_j^{-1} u_{\lambda_j}(x/R_j).$$

Then  $0 \leq v_j \leq 1$  is a bounded solution of

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla v_j) = 0 & \text{in } B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1} \\ v_j = \lambda_j \bar{u}_s & \text{on } \partial B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} (t^{1-2s} (v_j)_t) = \kappa_s v_j^p & \text{on } B_{R_j}^{n+1} \cap \{t = 0\}. \end{cases}$$

Moreover  $v_j \leq \bar{u}_s$  in  $B_{R_j}^{n+1} \cap \mathbb{R}_+^{n+1}$  and  $v_j(0) = 1$ . Using elliptic estimates we find (for a subsequence) that  $(v_j)$  converges uniformly on compact sets of  $\overline{\mathbb{R}_+^{n+1}}$  to a function  $v$  that is axially symmetric and solves

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla v) = 0 & \text{in } \mathbb{R}_+^{n+1} \\ -\lim_{t \rightarrow 0} (t^{1-2s} v_t) = \kappa_s v^p & \text{on } \mathbb{R}^n \times \{0\}. \end{cases}$$

Moreover  $0 \leq v \leq 1$ ,  $v(0) = 1$  and  $v \leq \bar{u}_s$ . This  $v$  restricted to  $\mathbb{R}^n \times \{0\}$  is a radial, bounded, smooth solution of (1.3) and from  $v \leq \bar{u}_s$  we deduce that  $v$  is stable.

**Acknowledgments:** L. Dupaigne is partially supported by Labex MILYON/ANR-10-LABX-0070. J. Dávila is supported by Fondecyt 1130360, CAPDE Millennium



Nucleus NC130017 and Fondo Basal CMM, Chile. The research of J. Wei is partially supported by NSERC of Canada.

## REFERENCES

- [1] Haïm Brezis, Thierry Cazenave, Yvan Martel, and Arthur Ramiandrisoa, *Blow up for  $u_t - \Delta u = g(u)$  revisited*, Adv. Differential Equations **1** (1996), no. 1, 73–90. MR1357955 (96i:35063)
- [2] Haim Brezis and Juan Luis Vázquez, *Blow-up solutions of some nonlinear elliptic problems*, Rev. Mat. Univ. Complut. Madrid **10** (1997), no. 2, 443–469. MR1605678 (99a:35081)
- [3] Xavier Cabré and Yannick Sire, *Nonlinear equations for fractional Laplacians, I: Regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), no. 1, 23–53, DOI 10.1016/j.anihpc.2013.02.001. MR3165278
- [4] Luis A. Caffarelli, Basilis Gidas, and Joel Spruck, *Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth*, Comm. Pure Appl. Math. **42** (1989), no. 3, 271–297, DOI 10.1002/cpa.3160420304. MR982351 (90c:35075)
- [5] Luis Caffarelli and Luis Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260, DOI 10.1080/03605300600987306. MR2354493 (2009k:35096)
- [6] L. Caffarelli, J.-M. Roquejoffre, and O. Savin, *Nonlocal minimal surfaces*, Comm. Pure Appl. Math. **63** (2010), no. 9, 1111–1144, DOI 10.1002/cpa.20331. MR2675483 (2011h:49057)
- [7] Antonio Capella, Juan Dávila, Louis Dupaigne, and Yannick Sire, *Regularity of radial extremal solutions for some non-local semilinear equations*, Comm. Partial Differential Equations **36** (2011), no. 8, 1353–1384, DOI 10.1080/03605302.2011.562954. MR2825595 (2012h:35361)
- [8] Wenxiong Chen, Congming Li, and Biao Ou, *Classification of solutions for an integral equation*, Comm. Pure Appl. Math. **59** (2006), no. 3, 330–343, DOI 10.1002/cpa.20116. MR2200258 (2006m:45007a)
- [9] M. Chipot, M. Chlebík, M. Fila, and I. Shafrir, *Existence of positive solutions of a semilinear elliptic equation in  $\mathbf{R}_+^n$  with a nonlinear boundary condition*, J. Math. Anal. Appl. **223** (1998), no. 2, 429–471, DOI 10.1006/jmaa.1998.5958. MR1629293 (99h:35060)
- [10] J. Dávila, *Singular solutions of semi-linear elliptic problems*, Handbook of differential equations: stationary partial differential equations. Vol. VI, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, pp. 83–176, DOI 10.1016/S1874-5733(08)80019-8. MR2569324 (2010k:35179)
- [11] Juan Dávila, Louis Dupaigne, and Alberto Farina, *Partial regularity of finite Morse index solutions to the Lane-Emden equation*, J. Funct. Anal. **261** (2011), no. 1, 218–232, DOI 10.1016/j.jfa.2010.12.028. MR2785899 (2012e:35090)
- [12] Juan Dávila, Louis Dupaigne, and Marcelo Montenegro, *The extremal solution of a boundary reaction problem*, Commun. Pure Appl. Anal. **7** (2008), no. 4, 795–817, DOI 10.3934/cpaa.2008.7.795.
- [13] Juan Dávila, Louis Dupaigne, Kelei Wang, and Juncheng Wei, *A Monotonicity Formula and a Liouville-type Theorem for a Fourth Order Supercritical Problem*, Advances in Mathematics, to appear.
- [14] Louis Dupaigne, *Stable solutions of elliptic partial differential equations*, Chapman & Hall/CRC Monographs and Surveys in Pure and Applied Mathematics, vol. 143, Chapman & Hall/CRC, Boca Raton, FL, 2011. MR2779463 (2012i:35002)
- [15] Eugene B. Fabes, Carlos E. Kenig, and Raul P. Serapioni, *The local regularity of solutions of degenerate elliptic equations*, Comm. Partial Differential Equations **7** (1982), no. 1, 77–116, DOI 10.1080/03605308208820218. MR643158 (84i:35070)
- [16] Mouhamed Moustapha Fall, *Semilinear elliptic equations for the fractional Laplacian with Hardy potential*. <http://arxiv.org/abs/1109.5530>
- [17] Mouhamed Moustapha Fall and Veronica Felli, *Unique continuation property and local asymptotics of solutions to fractional elliptic equations*. <http://arxiv.org/abs/1301.5119>
- [18] Alberto Farina, *On the classification of solutions of the Lane-Emden equation on unbounded domains of  $\mathbf{R}^N$* , J. Math. Pures Appl. (9) **87** (2007), no. 5, 537–561, DOI 10.1016/j.matpur.2007.03.001 (English, with English and French summaries). MR2322150 (2008c:35070)

- [19] Rupert L. Frank, Enno Lenzmann, and Luis Silvestre, *Uniqueness of radial solutions for the fractional laplacian*. <http://arxiv.org/abs/1302.2652>
- [20] B. Gidas and J. Spruck, *A priori bounds for positive solutions of nonlinear elliptic equations*, Comm. Partial Differential Equations **6** (1981), no. 8, 883–901, DOI 10.1080/03605308108820196. MR619749 (82h:35033)
- [21] Junichi Harada, *Positive solutions to the Laplace equation with nonlinear boundary conditions on the half space*. preprint
- [22] Ira W. Herbst, *Spectral theory of the operator  $(p^2 + m^2)^{1/2} - Ze^2/r$* , Comm. Math. Phys. **53** (1977), no. 3, 285–294. MR0436854 (55 #9790)
- [23] D. D. Joseph and T. S. Lundgren, *Quasilinear Dirichlet problems driven by positive sources*, Arch. Rational Mech. Anal. **49** (1972/73), 241–269.
- [24] Yan Yan Li, *Remark on some conformally invariant integral equations: the method of moving spheres*, J. Eur. Math. Soc. (JEMS) **6** (2004), no. 2, 153–180. MR2055032 (2005e:45007)
- [25] S. A. Molčanov and E. Ostrovskiĭ, *Symmetric stable processes as traces of degenerate diffusion processes.*, Teor. Veroyatnost. i Primenen. **14** (1969), 127–130 (Russian, with English summary). MR0247668 (40 #931)
- [26] Peter Poláčik, Pavol Quittner, and Philippe Souplet, *Singularity and decay estimates in superlinear problems via Liouville-type theorems. I. Elliptic equations and systems*, Duke Math. J. **139** (2007), no. 3, 555–579, DOI 10.1215/S0012-7094-07-13935-8. MR2350853 (2009b:35131)
- [27] Xavier Ros-Oton and Joaquim Serra, *The Pohozaev identity for the fractional Laplacian*. <http://arxiv.org/abs/1207.5986>
- [28] Frank Spitzer, *Some theorems concerning 2-dimensional Brownian motion*, Trans. Amer. Math. Soc. **87** (1958), 187–197. MR0104296 (21 #3051)
- [29] Kelei Wang, *Partial regularity of stable solutions to the supercritical equations and its applications*, Nonlinear Anal. **75** (2012), no. 13, 5238–5260, DOI 10.1016/j.na.2012.04.041. MR2927586

DEPARTAMENTO DE INGENIERÍA MATEMÁTICA AND CMM, UNIVERSIDAD DE CHILE, CASILLA 170 CORREO 3, SANTIAGO, CHILE  
*E-mail address:* jdavila@dim.uchile.cn

LAMFA, UMR CNRS 7352, UNIVERSITÉ DE PICARDIE JULES VERNE, 33 RUE ST LEU, 80039, AMIENS CEDEX, FRANCE  
*E-mail address:* louis.dupaigne@math.cnrs.fr

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF BRITISH COLUMBIA, VANCOUVER, B.C., CANADA, V6T 1Z2.  
*E-mail address:* jcwei@math.ubc.ca