HOMOTOPY EQUIVALENCE OF SHIFTED COTANGENT BUNDLES

RICARDO CAMPOS

Abstract. Given a bundle of chain complexes, the algebra of functions on its shifted cotangent bundle has a natural structure of a shifted Poisson algebra. We show that if two such bundles are homotopy equivalent, the corresponding Poisson algebras are homotopy equivalent.

We apply this result to L_∞-algebroids to show that two homotopy equivalent bundles have the same L_∞-algebroid structures.

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1. Introduction

The classical notion of a Lie algebroid consists of a vector bundle A over a manifold M together with a compatible Lie algebra structure on the space of sections $\Gamma(A)$ of A. More recently, due to the application of homotopy theoretical tools to theoretical physics [KS10, KS15] and to differential geometry (resolution of singular foliations) [Lav16, Lav16], there has been much interest in a derived version of Lie algebroids.

In the early 90's, T. Lada and J. Stasheff [LS93] introduced L_∞ algebras in the context of mathematical physics as a natural extension of differential graded Lie algebras, in which the Jacobi identity is only satisfied up to higher coherent homotopies given by multilinear brackets. The same approach of intertwining L_∞ algebras and manifolds give rise to the homotopical version of Lie algebroids, the so-called L_∞ algebroids [SSS12, Sev01].

It is often convenient to work in the dual setting of differential graded (dg) manifolds which are generalizations of smooth manifolds to higher geometry, in which spaces are locally modeled by chain complexes. We recall that in [Vor10] Voronov shows that given a graded vector bundle E, L_∞ algebroids over E are in one-to-one correspondence with non-positive dg manifold structures on E. Given this correspondence, we call E a *split graded manifold.*

Assume now that E dg vector bundle i.e., E is a sequence of vector bundles $(E_i)_{i\in\mathbb{Z}}$ endowed with a global differential $d\colon E_i \to E_{i+1}$ squaring to zero. The goal of this manuscript is to understand the behavior of the space L_∞ algebroid structures on E when we replace E by a homotopy equivalent split dg manifold F over the same base manifold M.

Our main result states that two homotopy equivalent split dg manifolds have essentially the same L_∞ algebroid structures.

Theorem 4.1 Let E and F be homotopy equivalent split dg manifolds concentrated in non-positive degrees. Then, there is a bijection

$$\left\{\begin{array}{l} L_\infty \text{ algebroid} \\ \text{structures on } E \end{array}\right\}\Big/\text{gauge eq.} \overset{1:1}{\longleftrightarrow} \left\{\begin{array}{l} L_\infty \text{ algebroid} \\ \text{structures on } F \end{array}\right\}\Big/\text{gauge eq.}$$

The crucial object to prove this result is the shifted cotangent bundle $T^*[1]E$ [Roy99]. The commutative algebra of functions of this space extends to a shifted Poisson algebra via the *big bracket* [KS04, Kra07, Vor10].

There is an of analog of Voronov's result stating that the space of L_∞ algebroid structures over E can be identified with the set of Maurer–Cartan elements of algebra of functions on $T^*[1]E$, the shifted cotangent bundle of E.

This prompts us to understand how the shifted cotangent bundle behaves under homotopy equivalence. Our other main result, in the form of Theorem 3.1 and Corollary 3.6, states that if E and F are two homotopy equivalent dg vector bundles, their algebras of functions are homotopy equivalent as Poisson algebras.

Theorem Let E and F be two homotopy equivalent split dg manifolds. Then, there exist $C^\infty(M)$-linear ∞-quasi-isomorphisms $\mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$ and $\mathcal{O}_{T^*[1]F} \rightsquigarrow \mathcal{O}_{T^*[1]E}$ of shifted Poisson algebras.

When E is concentrated in degree 0, L_∞ algebroids are precisely Lie algebroids, and 1-shifted Poisson structures are seen to be what is refereed to in the literature as quasi-Lie bialgebroids (see [Roy02b] for a definition of those and, e.g. [Ant08], for the description in terms of big bracket). Quasi-Lie bialgebroids are the infinitesimal version of quasi-Poisson groupoids (see [IPLGX12]).

The results that we present here are connected to the theory of shifted Poisson structures [CPT$^+$17]. In section 4.2 we see that under certain conditions our result allows us to conclude that two homotopy equivalent L_∞ algebroids have equivalent spaces of shifted Poisson structures. This matches recent advances by [BCSX17] and [Saf17].

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1.2. **Notation and conventions.** Throughout this manuscript the phrase *differential graded* or *dg* should be implicit everywhere. Concretely, unless otherwise explicit, a vector space V is a dg vector space (i.e. a cochain complex), Lie algebras are differential graded Lie algebras, locally ringed spaces are dg $\mathbb{R}$-algebras etc. We use cohomological conventions, i.e. all differentials have degree +1. In particular this means that taking linear duals reverses degrees $V_i^* = (V_{-i})^*$.

The vector spaces (such as the ones arising from dg manifolds) being considered are assumed to be finite dimensional in every degree and but not necessarily of bounded degree.

Given two differential graded vector spaces A and B, the induced differential on the space $\mathrm{Hom}(A, B)$ is the commutator, denoted by $[d, -]$, satisfying $[d, f] = f \circ d_A + (-1)^d d_B \circ f$, for $f \in \mathrm{Hom}(A, B)$ of degree d.

Finally, we consider the ground field to be $\mathbb{R}$ for concreteness but the reader will notice that all algebraic proofs hold over any field.

1.2.1. *Remark about degree shifts.* Given a vector space V, the notation $[k]$ denotes a shift of degree by k units, such that $(V[k])_i = V_{k+i}$. Throughout the text we will encounter algebraic structures whose operations are not in degree zero. Concretely, the functions on the shifted cotangent bundle form a 2-shifted Lie algebra or a $\mathsf{Lie}\{2\}$ algebra, a Lie algebra

whose Lie bracket has degree −2. When it is unambiguous, we might omit the shifts for simplicity.

As a precise definition one can define a $\mathsf{Lie}\{k\}$ algebra structure on V to be a Lie algebra structure on $V[-k]$. One should notice that this means that for odd k, a $\mathsf{Lie}\{k\}$ algebra has symmetric brackets, but when shifts are even, all formulas (and signs) stay the same.

We remark that one of the consequences of the degree shifts and the cohomological conventions is that on a $\mathsf{Lie}\{k\}$ algebra, a Maurer–Cartan element has degree 3.

2. Differential graded manifolds and the shifted cotangent bundle

In this section we intend to recall in detail the constructions and results associated to the shifted cotangent bundle of a split dg manifold. We recommend the texts [Fai17, Ant10, BP13] for a more thorough introduction to the topics of this section.

2.1. **Dg manifolds.**

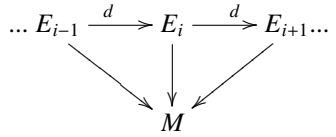
The origins of graded geometry and dg (differential graded) geometry can be traced back to physics, where ($\mathbb{Z}/2\mathbb{Z}$ graded) manifolds give for instance a proper treatment of ghosts in BRST deformation. Graded (resp. dg) manifolds are locally modeled by a graded (resp. dg) vector space V in the sense that a function on such a manifold is locally given by a function on the base manifold and a polynomial function on V.

Definition 2.1. *A graded manifold is a locally ringed space $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$, where the base M is a smooth manifold and around every point $x \in M$ there is an open set $U \ni x$ such that the structure sheaf $\mathcal{O}_{\mathcal{M}}(U) = C^\infty(U) \otimes S(V^*)$ is for some some graded vector space V.*

A dg manifold (also called a Q-manifold) a graded manifold equipped with a degree $+1$ cohomological vector field Q, i.e., a derivation of the algebra of functions such that $Q^2 = 0$.

In the present text we will be mostly interested in a subclass of dg manifolds that originate from bundles.

Example 2.2 (Dg vector bundles)**.** *Given a differential graded vector bundle E over M, i.e., a sequence of vector bundles $(E_i)_{i\in\mathbb{Z}}$ with differentials d*



such that $d^2 = 0$, one has a naturally associated dg manifold also denoted by E, given by its sheaf of sections $E = (M, \mathcal{O}_E = \Gamma(S(E^*)))$.

Notice that $d\colon E \to E$ induces a degree $+1$ map $Q\colon E^* \to E^* \hookrightarrow S(E^*)$ that extends to a square zero $C^\infty(M)$-linear derivation on $\Gamma(S(E^*))$.

Such dg manifolds are called split dg manifolds.

In fact, every non-negatively graded manifold originates from such a construction (Batchelor's theorem [Bat80]), even though the vector bundle E is non-canonically determined.

2.2. **Shifted cotangent bundle and the big bracket.**

Given a graded vector bundle $E \to M$, one can consider its shifted cotangent bundle $T^*[1]E = (M, \mathcal{O}_{T^*[1]E})$ (see [Roy02a, Ant10] for the constructions in the ungraded setting). Locally this space has coordinates $x^i \in M, \xi^a \in E, \underbrace{p_i \in TM, \theta_a \in E^*}_{\text{momentum coordinates}}$.

In these coordinates, the cohomological degree in the algebra $\mathcal{O}_{T^*[1]E}$ is given by $\deg(x^i) = 0$, $\deg(p_i) = 2$, $\deg(\xi^a) = d + 1$ for $\xi^a \in E_d$ and $\deg(\theta_a) = -d + 1$ for $\theta_a \in (E_d)^*$. We will also consider a biweight w on $\mathcal{O}_{T^*[1]E}$ compatible with the product, where $w(x^i) = (0, 0)$, $w(p_i) = (1, 1)$, $w(\xi^a) = (0, 1)$ and $w(\theta_a) = (1, 0)$.

Notice that there are natural global inclusions

$$C^\infty|_M \hookrightarrow \mathcal{O}_{T^*[1]E}$$
$$\Gamma(E) \hookrightarrow \mathcal{O}_{T^*[1]E}$$
$$\Gamma(E^*) \hookrightarrow \mathcal{O}_{T^*[1]E}$$

Remark 2.3. *Given the choices of a connections $\nabla_i \colon TM \to E$ one can also define a (non-canonical) inclusion $\Gamma(TM) \hookrightarrow \mathcal{O}_{T^*[1]E}$. With this choice one can identify*

$$\mathcal{O}_{T^*[1]E} \cong_\nabla S(TM[-2] \oplus E^*[-1] \oplus E[-1]).$$

Besides the commutative product, the space $\mathcal{O}_{T^*[1]E}$ has a natural Lie bracket $\{-,-\}$, the so-called *big bracket* [KS04, Roy02a] extending the natural pairing of E^* and E.

More concretely, the bracket has degree -2, biweight $(-1,-1)$ and it satisfies the following identities on generators

$$\{X, f\} = X \cdot f, \text{ for } X \in \Gamma(TM), f \in C^\infty|_M,$$
$$\{\epsilon, e\} = \langle \epsilon, e \rangle, \text{ for } e \in \Gamma(E), \epsilon \in \Gamma(E^*),$$

Even though the bracket is intrinsically defined, with the choice of a connection ∇ as in Remark 2.3 we also have $\{X, e\} = \nabla_X(e)$ and $\{X, \epsilon\} = \nabla_X(\epsilon)$.

The bracket is extended to the full algebra $\mathcal{O}_{T^*[1]E}$ by the Leibniz rule with respect to the product of functions, making $\mathcal{O}_{T^*[1]E}$ a shifted version of a Poisson algebra, also called a Pois_3 or e_3 algebra in the literature.

Remark 2.4. *Notice that since the differential has weight zero and the bracket has weight $(-1,-1)$, the (shifted) Poisson algebra $\mathcal{O}_{T^*[1]E}$ can be decomposed into a direct sum of (shifted) Lie algebras*

$$\mathcal{O}_{T^*[1]E} = \bigoplus_{k \geq 0} W_k,$$

where the Lie algebra $W_k = \bigoplus_{n \geq 0} W_{(n,n+k)}$ is spanned by all the elements whose biweights components have a common difference, i.e, elements of biweight $(0,k), (1,k+1), (2,k+2)$ and so on.

Suppose now that E was a dg vector bundle with differential d_E. It is easy to see that these constructions are compatible with the differential and that in this case $\mathcal{O}_{T^*[1]E}$ is a dg Poisson algebra. Another way to see this is that d_E a Maurer–Cartan element of $\mathcal{O}_{T^*[1]E}$, i.e. $\{d_E, d_E\} = 0$. Indeed, it follows from $d_E^2 = 0$ that $\{\{d_E, d_E\}, x\} = 0$ for every x element of E or E^*. Therefore, $\{d_E, d_E\}$ is central in $S(E \oplus E^*)$ but the center of this Lie algebra is $\mathbb{R}$ and therefore $\{d_E, d_E\} = 0$.

By twisting the space $\mathcal{O}_{T^*[1]E}$ by this Maurer–Cartan element, we recover a dg Poisson algebra structure on $\mathcal{O}^{d_E}_{T^*[1]E}$ that we will denote by $\mathcal{O}_{T^*[1]E}$ only.

2.3. **(Infinity) Algebroids.** The constructions from the previous section allow us to encode neatly some classical notions. For example, a Lie algebroid structure over M i.e., an ungraded Lie algebra bundle E with a compatible anchor map $\rho \colon E \to TM$, can be expressed as a solution of the Maurer–Cartan equation on $T^*[1]M$:

Proposition 2.5 ([Vai97, Roy99])**.** *Let M be a manifold and $E \to M$ a vector bundle concentrated in degree zero. The data of a Lie algebroid structure on E is equivalent to an element $\mu \in \mathcal{O}_{T^*[1]E}(M)$ of biweight $(1,2)$ such that $\{\mu, \mu\} = 0$.*

The identification is given by $\rho(X) \cdot f = \{\{X, \mu\}, f\}$ and $[X, Y] = \{\{X, \mu\}, Y\}$, for $X, Y \in \Gamma(TM, M)$ and $f \in C^\infty(M)$,

The same way the *homotopically correct* version of a Lie algebra is an L_∞ algebra, the notion of a Lie algebroid over a manifold M can be homotopically relaxed leading to the concept of an L_∞ algebroid. For technical reasons, for what follows we will suppose that all objects are non-positively graded.

Definition 2.6. *Let M be a smooth manifold and let $(E = (E_i)_{i\leq 0}, d)$ be a dg vector bundle over M concentrated in non-positive degree. An L_∞ algebroid structure on E is:*

(1) A dg bundle map $\rho\colon E \to TM$ called the anchor and
(2) A sequence of antisymmetric brackets $l_k = [\dots]_k\colon \Gamma(E^{\otimes k}) \to \Gamma(E)$ of degree $2-k$, for $k \geq 2$.

such that

(1) All brackets are $C^\infty(M)$ linear except the binary bracket if one of the entries is in degree 0. If that is the case, then it behaves as a vector field in the sense that if $X \in \Gamma(E_0)$ and $e \in \Gamma(E)$,

$$[X, fe]_2 = f[X, e]_2 + (\rho(X) \cdot f)e.$$

(2) The anchor intertwines l_2 and the bracket of vector fields

$$[\rho(x), \rho(y)] = \rho([x, y]), \forall x, y \in \Gamma(E_0).$$

(3) These brackets satisfy the structural axioms of an L_∞ algebra (3).

Remark 2.7. *Some authors such as* [Get10] *consider all brackets to be symmetric and of degree* 1 *(from an operadic perspective one would call these $L_\infty\{-1\}$ algebroids) while we follow conventions such as the ones of* [BP13]. *These are equivalent up to a degree shift of E.*

Analogous to Proposition 2.5 one can show that L_∞ algebroids are also given as solutions of the Maurer–Cartan equation.

Proposition 2.8 (Folklore)**.** *Let $E \to M$ be a split dg manifold concentrated in non-positive degrees, finite dimensional in every degree. The set of L_∞ algebroid structures over E is in biunivocal correspondence with the space of solutions of the Maurer–Cartan equation in $\mathcal{O}_{T^*[1]E}$ of biweight $(*, 1)$ such that the term in $E^* \otimes E = \mathrm{Hom}(E, E)$ is the differential $d\colon E \to E$.*

Sketch of proof. Due to the assumption of finite dimension, a map of bundles $E \to TM$ is equivalent to a section of $E^* \otimes TM$ and the data of the brackets corresponds to a section of $S(E^*) \otimes E$. The degree conditions imply that these correspond to elements of degree 3 in $\mathcal{O}_{T^*[1]E}$.

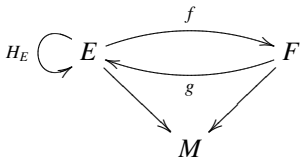
The bracket condition its easy to verify: The Maurer–Cartan equation can be split by left weight. On high weights we find the L_∞ structure equations and on left weight 2 the terms with the differential do not exist due to our degree restraints on E so the Maurer–Cartan equation gives us the same compatibility with the anchor as in the usual Lie algebroid case. □

Remark 2.9. *Some authors suppose that E is a graded manifold from the start and the L_∞ algebroid structure includes the datum of the differential d as a unary bracket l_1. The natural analog of the previous proposition holds, with the differential is recovered from the $E^* \otimes E$ part. If we recall that the differential d_E is itself a Maurer–Cartan element of $\mathcal{O}_{T^*[1]E}$, the two results are related from the general fact that if $\mathfrak{g}$ is a Lie algebra and $\mu \in \mathrm{MC}(\mathfrak{g})$, then $\nu \in \mathrm{MC}(\mathfrak{g}^\mu) \Leftrightarrow \nu + \mu \in \mathrm{MC}(\mathfrak{g})$.*

3. Proof of the main result

This section is devoted prove the main Theorem.

Theorem 3.1. *Let E and F be split dg manifolds over M. Suppose that there exist bundle maps $f\colon E \to F$ and $g\colon F \to E$ that are quasi-isomorphisms and a homotopy $H_E\colon E^\bullet \to E^{\bullet-1}$ such that $id_E - g \circ f = H_E d_E + d_E H_E$.*



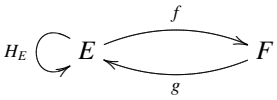
Then, here exists a $C^\infty(M)$ linear $L_\infty\{2\}$ quasi-isomorphism $\mathcal{U}\colon \mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$. Furthermore, this map:

- *is compatible with the symmetric algebra product,*
- *is compatible with the biweight in the sense that it preserves each component W_k from Remark 2.4 ($\mathcal{U}_n$ has biweight $(-n+1, -n+1)$),*
- *its first component $\mathcal{U}_1$ is the natural extension f and g^*.*

Precisely, by compatibility with the symmetric algebra product we mean that every $(\mathbb{U}_n)_{n\geq 2}$ acts as a derivation with respect to the map U_1. In particular, this means that $\mathcal{U}$ actually defines a weak equivalence of shifted Poisson algebras (an $\infty - \mathsf{Pois}_3$ algebra quasi-isomorphism).

3.1. **The case $M = *$.** In this section we prove the main theorem 3.1 over $M = *$ a point, which reduces to a problem in homotopical algebra. As we will see, this is the main part of the proof, as the formulas we will obtain over a point readily extend to a more general base.

In this case, E and F are just two dg vector spaces that are quasi-isomorphic with a prescribed homotopy.



The functions on the shifted cotangent bundle $T^*[1]E$ are given by the symmetric algebra $S(E[-1] \oplus E^*[-1])$.

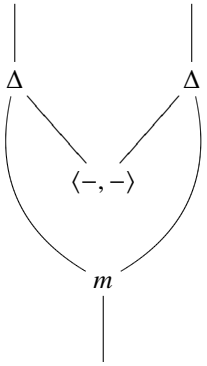
We define a map $\mathcal{U}_1\colon S(E[-1]\oplus E^*[-1]) \to S(F[-1]\oplus F^*[-1])$ by extending $f\colon E \to F$ and $g^*\colon E^* \to F^*$ to a map of commutative algebras.

Recall that given a dg vector space V, the space $S(V)$ admits a bialgebra structure given by the canonical coproduct $\Delta\colon S(V) \to S(V) \otimes S(V)$ by

$$\Delta(v_1 \dots v_n) = \sum_{\substack{p\leq n \\ \sigma\in \mathbb{S}_n}} \pm v_{\sigma^{-1}(1)} \dots v_{\sigma^{-1}(p)} \otimes v_{\sigma^{-1}(p+1)} \dots v_{\sigma^{-1}(n)}.$$

Notice that under this description, the Poisson bracket on $S(E[-1] \oplus E^*[-1]) = \mathcal{O}_{T^*[1]E}$ has the following nice form

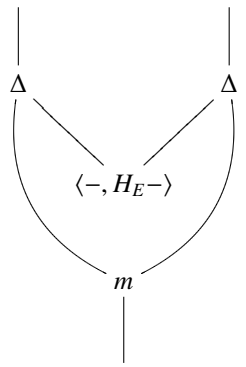
$l_{T^*[1]E} \coloneqq \{-,-\} =$



,

where m stands for the multiplication (in the symmetric algebra) and $\langle -,- \rangle$ denotes the pairing between E^* and E being zero otherwise. By convention, elements of E^* will be placed on the first entry of $\langle -,- \rangle$ and elements of E will be placed on the second entry.

We define the operator $\mathcal{R}_2 \coloneqq \mathcal{O}_{T^*[1]E} \otimes \mathcal{O}_{T^*[1]E} \to \mathcal{O}_{T^*[1]E}$



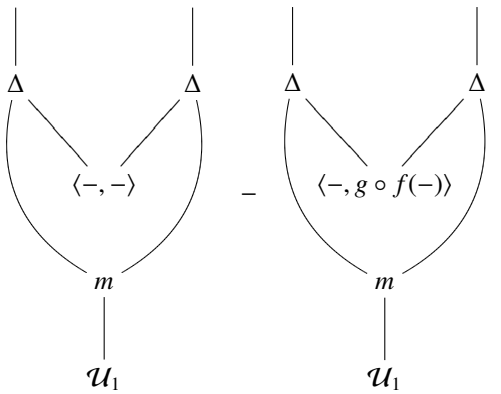
,

Finally, we define $\mathcal{U}_2 \colon \mathcal{O}_{T^*[1]E} \to \mathcal{O}_{T^*[1]F}$ to be $\mathcal{U}_2 \coloneqq \mathcal{U}_1 \circ \mathcal{R}_2$.

Notice that besides the homotopy, all the operations involved in $\mathcal{U}_2$ commute with the differentials, from which it follows that

(1)

$[d, \mathcal{U}_2] =$



$=: \mathcal{U}_1 \circ l_{T^*[1]E} - \mathcal{U}_1 \circ \tilde{l}_{T^*[1]E}$

and a similar formula without the terms $\mathcal{U}_1$ holds if we replace $\mathcal{U}_2$ by $\mathcal{R}_2$. One can easily check on generators that the second term of the equation, $\mathcal{U}_1 \circ \tilde{l}_{T^*[1]E}$ is equal to $\{\mathcal{U}_1(-), \mathcal{U}_1(-)\}_{T^*[1]E}$ from where the case $n = 2$ from equation (4) follows.

To define the higher components of the L_∞ morphism we require some set-up. Let Tree_n be the set of trees with n labeled vertices. To an element $T \in \mathrm{Tree}_n$ one can associate a map $T\colon O^{\otimes n}_{T^*[1]E} \to O_{T^*[1]E}$ of degree $n-1$. The value of $T(x_1, x_2, \dots, x_n)$ is obtained in the following way:

Let $e_1, \dots, e_{n-1}$ be the set of edges of T and consider a choice of $2n-2$ elements $\alpha_1, \beta_1, \dots, \alpha_{n-1}, \beta_{n-1}$ each one of that from either E or E^* such that:

- For every k, if e_k connects vertices i and j, α_k is a factor of x_i and β_k is a factor of x_j,
- There is no repetition of choices.

Given such a choice one can consider the product $\langle \alpha_1, H(\beta_1)\rangle \dots \langle \alpha_{n-1}, H(\beta_{n-1})\rangle \widehat{x_1 x_2 \dots x_n}$, where $\widehat{x_1 x_2 \dots x_n}$ denotes the product of all x_i's but with our choice of α's and β's removed[1] together with the appropriate Koszul sign corresponding to the elements removed.

Finally, the value of $T(x_1, x_2, \dots, x_n) = \sum_{\substack{\text{choices of} \\ \alpha_1, \dots \beta_{n-1}}} \langle \alpha_1, H(\beta_1)\rangle \dots \langle \alpha_{n-1}, H(\beta_{n-1})\rangle \widehat{x_1 x_2 \dots x_n}$ is obtained by summing over all possible choices the products described.

Heuristically, every edge of a tree corresponds to an application of $\mathcal{R}_2$ and indeed $\mathcal{R}_2 =$  $\in \mathrm{Tree}_2$.

Remark 3.2. *Notice that some choices regarding the ordering and orientation of edges of T has to be done to compute $T(x_1, x_2, \dots, x_n)$. Since the target $O_{T^*[1]E}$ is commutative, all choices lead to the same result up to a sign.*

We fix the convention that edges are oriented from the smaller vertex to the bigger vertex and the ordering of edges is done by comparing the smaller label and then the bigger label.

In particular it follows that natural the action of $\mathbb{S}_n$ permuting the labels of the vertices produces signs.

For all $n \geq 1$ we define the operators $\mathcal{R}_n\colon O^{\otimes n}_{T^*[1]E} \to O_{T^*[1]E}$ of degree $n-1$ as $\mathcal{R}_n = \sum_{T \in \mathrm{Tree}_n} T$. Notice that this definition gives $\mathcal{R}_1 = \mathrm{id}_{O_{T^*[1]E}}$.

We also define $\mathcal{U}_n \coloneqq \mathcal{U}_1 \circ \mathcal{R}_n \circ O^{\otimes n}_{T^*[1]E} \to O_{T^*[1]F}$.[2]

Proposition 3.3. *The maps $\mathcal{R}_n$ defined above satisfy the following equations, for all $n \geq 2$:*

$$[d, \mathcal{R}_n] = \sum_{\sigma \in \mathrm{Sh}^{-1}_{2,n-2}} \mathrm{sgn}(\sigma)(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\sigma - \sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}^{-1}_{p,q}}} \mathrm{sgn}(\sigma)(-1)^{p-1} \tilde{l}_{T^*[1]E} \circ (\mathcal{R}_p, \mathcal{R}_q)^\sigma \tag{2}$$

where $\tilde{l}_{T^*[1]E}$ is the twisted bracket defined in equation (1).

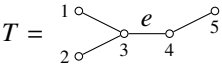
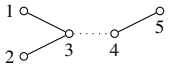
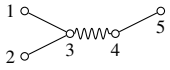
Before proving this proposition notice that by composing the equations above with $\mathcal{U}_1$ and using the observation that $\mathcal{U}_1 \circ \tilde{l}_{T^*[1]E} = l_{T^*[1]F} \circ (\mathcal{U}_1, \mathcal{U}_1)$ we recover exactly the equations (4) defining an L_∞ morphism.

Corollary 3.4. *The maps $\mathcal{U}_n$ defined above form an L_∞ algebra morphism.*

Proof of Proposition 3.3. Given a tree $T \in \mathrm{Tree}_n$ and e an edge of T, we denote by T^e the same tree T but with the edge e replaced by a dashed edge. Similarly, we denote by $T^{\sim e}$ the same edge e replaced by a wavy edge instead.

[1] Keep in mind the convention that $\langle a, b\rangle$ is zero unless one of a, b is in E and the other one in E^*.

[2] We bring attention to the fact that there is some similarity of the formulas for $\mathcal{U}_n$ and the ones appearing in the homotopy transfer theorem [Val14, LV12].

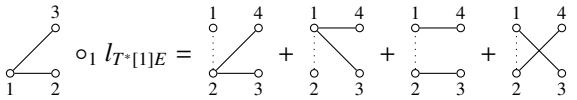
$T =$  , $T^e =$  , $T^{\sim e} =$ 

For $T \in \mathrm{Tree}_n$ we define an action of these modified trees, $T^e, T^{\sim e} \colon \mathcal{O}^{\otimes n}_{T^*[1]E} \to \mathcal{O}_{T^*[1]E}$ of degree $n-2$ by the same formula as T, except that on the action of the edge corresponding to e connecting vertices i and j, with T^e we perform the pairing $\langle \alpha_i, \beta_j \rangle$ and with $T^{\sim e}$ we perform the twisted pairing $\langle \alpha_i, g \circ f(\beta_j) \rangle$.

Notice that since the commutator with the differential $[d, -]$ acts by derivations, the computation of $[d, \mathcal{R}_n]$ produces the same kind of composition, except that it replaces one instance of $\langle -, H- \rangle$ by $\langle -, - \rangle - \langle -, g \circ f - \rangle$, just as in equation (1). In terms of trees, we have that $[d, T] = \sum_{e \text{ edge}} T^e - T^{\sim e}$, so we can also interpret $[d, \mathcal{R}_n]$ as a sum of all possible trees of n vertices with a dotted edge, minus a sum of all trees with n vertices with a wavy edge.

We claim that the summands corresponding to the terms T^e correspond to the terms $\sum_{\sigma \in \mathrm{Sh}^{-1}_{2,n-2}} \mathrm{sgn}(\sigma)(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\sigma$.

This follows from the observation that given a tree $\Gamma \in \mathrm{Tree}_{n-1}$, the operation $T(\{x_1, x_2\}, x_3, \ldots, x_n)$ can be expressed as a sum of trees with a dotted edge. Concretely, as a quick inspection shows, $\Gamma \circ_1 l_{T^*[1]E}$ is obtained by inserting an graph  on the vertex labeled by 1 of Γ and summing over all possible ways (i.e. $2^{\text{valence of 1}}$) of reconnecting the incident edges, followed by a shift by 1 of all other labels.



This allows us to conclude that all terms of $(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\sigma$ are dotted trees. We just need to show that every dotted tree appears exactly once on the sum over all $(2, n-2)$ unshuffles.

Let us consider an arbitrary dotted tree T^e, where $T \in \mathrm{Tree}_n$. Suppose that e connects vertices $i < j$. There is a unique unshuffle $\tau \in Sh^{-1}_{2,n-2}$ sending i to 1 and j to 2. It is then clear that only $(\mathcal{R}_{n-1} \circ_1 l_{T^*[1]E})^\tau$ produces trees with a dotted edge connecting vertices i and j. Conversely, if we denote by $T/e \in \mathrm{Tree}_{n-1}$ be the graph obtained by the contraction of the edge e one sees that we recover T^e from the insertion of $l_{T^*[1]E}$ in T/e.[3]

To finish the proof it remains to show that

$$\sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}^{-1}_{p,q}}} \mathrm{sgn}(\sigma)(-1)^{p-1} \tilde{l}_{T^*[1]E} \circ (\mathcal{R}_p, \mathcal{R}_q)^\sigma = \sum_{T \in \mathrm{Tree}_n} \sum_{e \text{ edge of } T} T^{\sim e}.$$

The proof is analogous to the other case. We start by noting that for $T_p \in \mathrm{Tree}_p$ and $T_q \in \mathrm{Tree}_q$, $\tilde{l}_{T^*[1]E} \circ (T_i, T_q)$ is obtained summing over all possible ways $(p \times q)$ of connecting T_p and T_q with a wavy edge, and shifting the labels of T_q up by p units. It follows that $\tilde{l}_{T^*[1]E} \circ (\mathcal{R}_p, \mathcal{R}_q)^\sigma$ is a sum of elements of the form $T^{\sim e}$, where $T \in \mathrm{Tree}_n$. To see that every tree appears exactly once, one just notices that given a tree with a wavy edge, removing the wavy edge results in a disconnected graph composed of two trees, one in Tree_p and the other one in Tree_{n-p} whose labels are uniquely retained by an element of $\mathrm{Sh}^{-1}_{p,q}$. □

3.2. **The global case.** Suppose now that E and F are split dg manifolds over an arbitrary manifold M with maps f, g and a homotopy H_E as before. The two maps f and g induce a map $\mathcal{U}_1 \colon \mathcal{O}_{T^*[1]E} \to \mathcal{O}_{T^*[1]F}$ of commutative algebras by extending the maps $f \colon E \to F$,

[3]Notice that there is an appearance of a sign factor $\mathrm{sgn}(\tau)$ due to the considerations from Remark 3.2.

$g^*\colon E^* \to F^*$, $\mathrm{id}\colon\colon \Gamma(TM, M) \subset \mathcal{O}_{T^*[1]E} \to \Gamma(TM, M) \subset \mathcal{O}_{T^*[1]F}$[4] and $\mathrm{id}\colon\colon C^\infty(M) \subset \mathcal{O}_{T^*[1]E} \to C^\infty(M) \subset \mathcal{O}_{T^*[1]F}$.

It is easy to see that $\mathcal{U}_1$ intertwines the Lie brackets on $\mathcal{O}_{T^*[1]E}$ and $\mathcal{O}_{T^*[1]F}$ whenever at least one of the elements being bracketed does not belong to $S(E \oplus E^*)$.

We note that the formulas used for $\mathcal{U}_n, n \geq 2$ can be defined over any manifold, since the homotopy H_E is globaly defined on the bundle E. It follows that the natural extension of $\mathcal{U}_n$ give a well defined L_∞ quasi-isomorphism $\mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$:

$$\mathcal{U}_n(x_1, \dots, x_n) \coloneqq \begin{cases} \text{same formula as before if all } x_i \in S(E \oplus E^*) \\ 0 \text{ otherwise.} \end{cases}$$

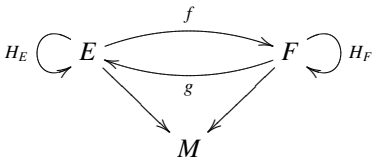
3.3. **Remarks about the hypothesis of homotopy equivalence.** The reader might be surprised that it seems that we almost did not use F at all in the proofs in this section, by reducing the problem to work with $\mathcal{R}_n$ instead of $\mathcal{U}_n$.

The reason for this is that one can consider a "twisted shifted cotangent bundle" $\widetilde{\mathcal{O}_{T^*[1]E}}$ given by the same base space space but with the twisted Lie bracket $\tilde{l}_{T^*[1]E}$. What we have shown is that $(\mathcal{R}_n)_{n>1}$ realise an L_∞ isomorphism $\mathcal{O}_{T^*[1]E} \to \widetilde{\mathcal{O}_{T^*[1]E}}$ extending the identity map. The result follows from the fact that $\mathcal{U}_1$ defines a strict Lie quasi-isomorphism $\widetilde{\mathcal{O}_{T^*[1]E}} \to \mathcal{O}_{T^*[1]F}$.

However, over manifolds M different from a point, the author advises against saying that $T^*[1]E$ and $T^*[1]F$ are homotopy equivalent as locally ringed spaces. The reason for this is that $\mathcal{O}_{T^*[1]E}$ and $\mathcal{O}_{T^*[1]F}$ are indeed homotopy equivalent as $\mathbb{R}$-Lie algebras, since quasi-isomorphisms are (quasi-)invertible over $\mathbb{R}$, but that is not necessarily the case over $C^\infty(M)$.

This problem can be circumvented if we suppose E and F are homotopic.

Definition 3.5. *Two split dg manifolds E and F over M are homotopy equivalent if there exist bundle maps $f\colon E \to F$ and $g\colon F \to E$ and global homotopies $H_E\colon E^\bullet \to E^{\bullet+1}$ and $H_F\colon F^\bullet \to F^{\bullet+1}$ such that $id_E - g \circ f = H_E d_E + d_E H_E$ and $id_F - f \circ g = H_F d_F + d_F H_F$*



Applying Theorem 3.1 on both directions we conclude that homotopy equivalent split dg manifolds have indeed homotopy equivalent shifted tangent bundles.

Corollary 3.6. *Let E and F be homotopy equivalent split dg manifolds. Then, $T^*[1]E$ and $T^*[1]F$ are homotopy equivalent in the sense that there exist $L_\infty\{2\}$ quasi-isomorphisms $\mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$ and $\mathcal{O}_{T^*[1]F} \rightsquigarrow \mathcal{O}_{T^*[1]E}$ over $C^\infty(M)$ inverse to each other.*

Moreover, these maps preserve the biweights.

4. Applications

4.1. L_∞ **algebroids.** Let E and F be non-positively graded split dg manifolds and maps f,g and H_E such as in the conditions of the main Theorem 3.1.

Recall from Proposition 2.8 that L_∞ algebroid structures over E are the same as Maurer–Cartan elements of $\mathcal{O}_{T^*[1]E}(M)$ of biweight $(*, 1)$. Since $\mathcal{U}_n$ has biweight $(-n+1, -n+1)$, it sends n elements of biweight $(*, 1)$ to an element of biweight $(*, 1)$. It follows that $\mathcal{U}\colon \mathcal{O}_{T^*[1]E} \rightsquigarrow \mathcal{O}_{T^*[1]F}$ maps Maurer–Cartan elements of biweight $(*, 1)$ to Maurer–Cartan elements of biweight $(*, 1)$. It follows from the Goldman–Millson Theorem A.7 and the main Theorem 3.1 that E and F have the same L_∞ algebroid structures.

[4]Strictly speaking, these inclusions depend on a choice, but clearly the map is canonically defined.

Theorem 4.1. *Let E and F be split dg manifolds concentrated in non-positive degrees that are homotopy equivalent. Then, there is a bijection*

$$\left\{ \begin{matrix} L_\infty \text{ algebroid} \\ \text{structures on } E \end{matrix} \right\} \Big/ \text{gauge eq.} \overset{1:1}{\longleftrightarrow} \left\{ \begin{matrix} L_\infty \text{ algebroid} \\ \text{structures on } F \end{matrix} \right\} \Big/ \text{gauge eq.}$$

4.2. **Shifted Poisson structures.** The results that we present here are certainly connected to the theory of shifted Poisson structures [CPT+17], see also [BCSX17, Saf17].

Let E be a split dg manifold and (E, ϕ^E) an L_∞ algebroid structure on E, i.e., ϕ^E is a Maurer–Cartan element of $\mathcal{O}_{T^*[1]E}$, as in the previous section. One can then twist the Poisson algebra $\mathcal{O}_{T^*[1]E}$ by ϕ^E the Lie algebra $\mathcal{O}^{\phi^E}_{T^*[1]E} = (\mathcal{O}_{T^*[1]E}, d^E + \{\phi^E, -\}, \{-,-\})$. We propose the following definition of a 1-shifted Poisson structure.

Definition 4.2. *A 1-shifted Poisson structure over the L_∞ algebroid (E, ϕ^E) is a Maurer–Cartan element in $\mathcal{O}^{\phi^E}_{T^*[1]E}$ of biweight $(*, \geq 2)$.*

From Corollary 3.6 and Lemma A.5 we obtain the following result.

Corollary 4.3. *Let E and F be homotopy equivalent split dg manifolds. Suppose that ϕ^E and ϕ^F are L_∞ algebroid structures on E and F respectively, such that the map $\mathcal{U}$ constructed in Theorem 3.1 satisfies $\mathcal{U}(\phi^E) = \phi^F$. Then, E and F have the same 1-shifted Poisson structures up to gauge equivalence.*

Appendix A. Recollections about L_∞ algebras and Maurer–Cartan elements

In this Appendix we recall some of the classical homotopy theory of Lie algebras and their Maurer–Cartan elements that are used in this paper. We assume that the Lie algebras are unshifted, i.e., the bracket has degree zero, but all statements hold for shifted Lie algebras c.f. Section 1.2.1.

Recall that an L_∞ algebra structure on the differential graded vector space (A, d) is a family of multilinear antisymmetric maps (the multibrackets) $[-, \dots, -] = l_n \colon A^{\otimes n} \to A$ of degree $|l_n| = 2 - n$ for $n \geq 2$ satisfying the higher Jacobi identities:

$$\sum_{\substack{p+q=n+1 \\ p,q>1}} \sum_{\sigma \in \mathrm{Sh}^{-1}_{q,p-1}} \mathrm{sgn}(\sigma)(-1)^{(p-1)q}(l_p \circ_1 l_q)^\sigma = [d, l_n], \tag{3}$$

where $\mathrm{Sh}^{-1}_{q,p-1} \subset \mathbb{S}_{q+p-1}$ denotes the $(q, p-1)$ unshuffles.

Most results in this section can be generalized to L_∞ algebras but since they are not necessary for us they are stated in terms of Lie algebras and L_∞ morphisms for simplicity of formulas.

Definition A.1. *An L_∞ morphism $\mathcal{U} \colon A \rightsquigarrow B$ between two Lie algebras (A, l_A, d_A) and (B, l_B, d_B) is a sequence of maps $\mathcal{U}_n \colon S^n A \to B, \forall n \geq 1$ of degree $1 - n$ such that $\mathcal{U}_1$ commutes with the differentials, i.e $[d, \mathcal{U}_1] = 0$ and*

$$[d, \mathcal{U}_n] = \sum_{\sigma \in \mathrm{Sh}^{-1}_{2,n-2}} \mathrm{sgn}(\sigma)(\mathcal{U}_{n-1} \circ_1 l_A)^\sigma - \sum_{\substack{p+q=n \\ \sigma \in \mathrm{Sh}^{-1}_{p,q}}} \mathrm{sgn}(\sigma)(-1)^{p-1} l_B \circ (\mathcal{U}_p, \mathcal{U}_q)^\sigma \tag{4}$$

Definition A.2. *Let $\mathfrak{g}$ be a differential graded Lie algebra. We say that a degree 1 element $\mu \in \mathfrak{g}^1$ is a Maurer–Cartan element if it satisfies the quadratic equation*

$$d\mu + \frac{1}{2}[\mu, \mu] = 0$$

The set of Maurer–Cartan elements of a Lie algebra $\mathfrak{g}$ is denoted by $\mathrm{MC}(\mathfrak{g})$.

Definition A.3. *A filtered Lie algebra is a Lie algebra* $\mathfrak{g}$ *equipped with a complete descending filtration* $\mathcal{F}^\bullet$ *of Lie algebras i.e.* $\mathfrak{g} = \mathcal{F}^1\mathfrak{g} \supset \mathcal{F}^2\mathfrak{g} \supset \mathcal{F}^2\mathfrak{g} \supset \dots$ *satisfying* $\left[\mathcal{F}^i\mathfrak{g}, \mathcal{F}^j\mathfrak{g}\right] \subset \mathcal{F}^{i+j}\mathfrak{g}$*, such that* $\mathfrak{g}$ *is complete with respect to this filtration*

$$\mathfrak{g} = \varprojlim_k \mathfrak{g}/\mathcal{F}^k\mathfrak{g}.$$

Let $\mathfrak{g}$ and $\mathfrak{h}$ be filtered Lie algebras. It is easy to check that given an L_∞ morphism $\mathcal{U} = (\mathcal{U}_k)_{k\geq 1} \colon \mathfrak{g} \rightsquigarrow \mathfrak{h}$ compatible with the filtrations[5] and a Maurer-Cartan element $\mu \in \mathfrak{g}$, then, the element

$$\mathcal{U}(\mu) \coloneqq \sum_{n=1}^{\infty} U_n(\mu, \dots, \mu) \in \varprojlim \mathfrak{h}/\mathcal{F}^k\mathfrak{h} = \mathfrak{h} \tag{5}$$

is a Maurer-Cartan element of $\mathfrak{h}$.

Given a Maurer-Cartan element μ of a Lie algebra $\mathfrak{g}$ one often considers the corresponding twisted Lie algebra $\mathfrak{g}^\mu$.

Definition A.4. *Let* $\mathfrak{g}$ *be a Lie algebra and* $\mu \in \mathrm{MC}(\mathfrak{g})$*. The Lie algebra* $\mathfrak{g}$ *twisted by* μ *is a Lie algebra denoted by* $\mathfrak{g}^\mu$*, that is equal to* $\mathfrak{g}$ *as a graded vector space, with the same Lie bracket* $[-,-]_\mathfrak{g} = [-,-]_{\mathfrak{g}^\mu}$ *but with a twisted differential given by*

$$d_{\mathfrak{g}^\mu} = d_\mathfrak{g} + [\mu, -].$$

Twisting is a homotopically stable property. The following result follows from a simple spectral sequence argument.

Proposition A.5 ([Dol05], Proposition 1)**.** *Let* $\mathfrak{g}$ *and* $\mathfrak{h}$ *be Lie algebras and* $U \colon \mathfrak{g} \to \mathfrak{h}$ *be an* L_∞ *morphism.*

If for all k*,* $U_1 \colon \mathcal{F}^k\mathfrak{g} \to \mathcal{F}^k\mathfrak{h}$ *is a quasi-isomorphism, then for any* $\mu \in \mathrm{MC}(\mathfrak{g})$*, the induced map* $\mathcal{U}^\mu \colon \mathfrak{g}^\mu \rightsquigarrow \mathfrak{h}^{\mathcal{U}(\mu)}$ *is an* L_∞ *quasi-isomorphism.*

Given a Lie algebra $\mathfrak{g}$ and an commutative algebra A, the space $\mathfrak{g} \otimes A$ inherits a natural Lie algebra structure by declaring the bracket to be A-bilinear, i.e., $[X \otimes a, X' \otimes a'] = [X, X'] \otimes aa'$. In the concrete case of the polynomial forms $A = \Omega_{\mathrm{poly}}([0,1]) = \mathbb{K}[t, dt]$, we get a natural Lie algebra structure on $\mathfrak{g}[t, dt]$.

Definition A.6. *Let* $\mathfrak{g}$ *be a Lie algebra. Two Maurer–Cartan elements* $\mu_0, \mu_1 \in \mathrm{MC}(\mathfrak{g})$ *are said to be gauge equivalent if there is a Maurer–Cartan element* $\mu_t \in \mathfrak{g}[t, dt]$ *interpolating* μ_0 *and* μ_1*.*

This definition amounts to say that μ_t can be written as

$$\mu_t = m_t + h_t dt$$

where m_t can be understood as a family of Maurer-Cartan elements in $\mathfrak{g}$, connected by a family of infinitesimal homotopies (gauge transformations) h_t. The Maurer–Cartan equation for μ_t translates into the two equations

$$dm_t + \frac{1}{2}[m_t, m_t] = 0, \qquad \dot{m}_t \frac{\partial m_t}{\partial t} + dh_t + [h_t, m_t] = 0.$$

Remarkably, the Goldman–Millson theorem states that under appropriate conditions one can identify the Maurer–Cartan spaces of quasi-isomorphic Lie algebras.

Theorem A.7 (Goldman–Millson [DR15])**.** *Let* $U \colon \mathfrak{g} \to \mathfrak{h}$ *be an* L_∞ *morphism of filtered Lie algebras. Suppose furthermore that on the associated graded level the map* $\mathrm{gr}\, U \colon \mathrm{gr}\,\mathfrak{g} = \bigoplus \mathcal{F}^\bullet\mathfrak{g}/\mathcal{F}^{\bullet+1}\mathfrak{g} \to \mathrm{gr}\,\mathfrak{h}$ *is a quasi-isomorphism. Then, formula 5 induces a bijection*

$$U \colon \mathrm{MC}(\mathfrak{g})/\textit{gauge equiv.} \to \mathrm{MC}(\mathfrak{h})/\textit{gauge equiv.}$$

[5] In the sense that $\mathcal{U}_k(\mathcal{F}^{i_1}\mathfrak{g}, \dots, F^{i_k}\mathfrak{g}) \subset F^{i_1+\dots+i_k}\mathfrak{h}$.

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Laboratoire Analyse, Géométrie et Applications, Université Paris 13, Sorbonne Paris Cité, CNRS, UMR 7539, 93430 Villetaneuse, France. Email: campos@math.univ-paris13.fr