Exact integration for products of power of barycentric coordinates over $d$-simplexes in $\mathbb{R}^n$

Francois Cuvelier

To cite this version:
Francois Cuvelier. Exact integration for products of power of barycentric coordinates over $d$-simplexes in $\mathbb{R}^n$. 2018. hal-01816347

HAL Id: hal-01816347
https://sorbonne-paris-nord.hal.science/hal-01816347
Preprint submitted on 15 Jun 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
Exact integration for products of power of barycentric coordinates over \(d\)-simplexes in \(\mathbb{R}^n\)

François Cuvelier∗

2018/06/15

Abstract

Exact integral computation over a \(d\)-simplex in \(\mathbb{R}^n\) for products of powers of its barycentric coordinates is done in [9] by using mathematical induction and coordinate mappings. In this note we give a new proof using Laplace transformations without mathematical induction.

Contents

1 Notations and definitions 2

2 Some results on the unit \(d\)-simplex 3

2.1 unit \(d\)-simplex volume ........................................ 3

2.2 Magic formula .................................................. 5

3 Some results on a \(d\)-simplex in \(\mathbb{R}^n\) 6

3.1 Gradients of Barycentric coordinates on a \(d\)-simplex .......... 6

3.2 Integration over a \(d\)-simplex .................................. 8

3.3 Volume of a \(d\)-simplex ........................................ 11

3.4 Magic formula .................................................. 11

∗Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS UMR 7539, 99 Avenue J-B Clément, F-93430 Villetaneuse, France, cuvelier@math.univ-paris13.fr

This work was supported in part by the ANR project DEDALES under grant ANR-14-CE23-0006.
Local shape functions of a large variety of finite element on a d-simplex \( K \subset \mathbb{R}^n \) can be expressed in function of the barycentric coordinates \( \{\lambda_0, \ldots, \lambda_d\} \) of \( K \) and their derivatives (see [1] for examples).

In [9], the authors give a proof of the magic formula: let \( \nu = (\nu_0, \ldots, \nu_d) \in \mathbb{N}^{d+1} \), then

\[
\int_{K} \prod_{i=0}^{d} \lambda_{\nu_i}(q) dq = d! \frac{\prod_{i=0}^{d} \nu_i!}{(d + \sum_{i=0}^{d} \nu_i)!}
\]

(1)

where \( |K| \) is the volume of \( K \). In their proof, mathematical induction and coordinate mappings are mainly used. In this note we give a new proof of this formula using Laplace transformations without mathematical induction.

Firstly we recall definitions of a d-simplex in \( \mathbb{R}^n \) and of its barycentric coordinates. Thereafter we introduce Laplace transforms to compute the volume of the unit d-simplex \( \hat{K} \subset \mathbb{R}^d \) and the magic formula (1) over \( \hat{K} \). In the last section, we propose to compute the gradients of the barycentric coordinates by solving linear systems. We also present the mapping of an integral over a d-simplex in \( \mathbb{R}^n \) to the reference unit d-simplex, allowing to prove (1).

1 Notations and definitions

Let \( n \in \mathbb{N}^* \) be the space dimension and \( d \in [0, n] \). We recall the definition of a d-simplex in \( \mathbb{R}^n \) as well as its barycentric coordinates.

**Definition 1 (d-simplex)** A d-simplex \( K \subset \mathbb{R}^n \) is the convex hull of \((d + 1)\) points \( q^0, \ldots, q^d \) of \( \mathbb{R}^n \) which form the vertices of \( K \).

\[
K = \left\{ q \in \mathbb{R}^n \mid q = \sum_{i=0}^{d} \theta_i q^i, \ \text{with} \ \forall i \in [0, d], \ \theta_i \geq 0, \ \text{and} \ \sum_{i=0}^{d} \theta_i = 1 \right\}.
\]

(2)

For example, a 2-simplex is a triangle and a 3-simplex is a tetrahedron. It will be always assumed that a d-simplex is not degenerated, i.e., the set of vectors \( \{q^i - q^0\}_{i=1}^{d} \) is linearly independent.

**Definition 2 (Barycentric coordinates)** Let \( K \subset \mathbb{R}^n \) be a non-degenerate d-simplex and \( \{q^i\}_{i=0}^{d} \) its vertices. The parametrization of \( K \) with a convex combination of the vertices reads as follows

\[
K = \left\{ q \in \mathbb{R}^n \mid q = \sum_{i=0}^{d} \lambda_i(q) q^i, \ \text{with} \ \forall i \in [0, d], \ \lambda_i(q) \geq 0, \ \text{and} \ \sum_{i=0}^{d} \lambda_i(q) = 1 \right\}.
\]

(3)

The coefficients \( \lambda_0(q), \ldots, \lambda_d(q) \) are called the barycentric coordinates on \( K \) of \( q \).

As immediate property, the barycentric coordinates on \( K \) satisfy

\[
\lambda_i(q^j) = \delta_{i,j}, \ \forall (i, j) \in [0, d].
\]

(4)
2 Some results on the unit d-simplex

The unit d-simplex \( \tilde{K}^d \subset \mathbb{R}^d \) is defined by the \( d + 1 \) vertices
\[
\{ \hat{q}^0, \hat{q}^1, \ldots, \hat{q}^d \} = \{0, \hat{e}^1, \ldots, \hat{e}^d \}
\]
where \( \{\hat{e}^1, \ldots, \hat{e}^d\} \) is the standard basis of \( \mathbb{R}^d \). We have
\[
\tilde{K}^d = \left\{ \hat{q} \in \mathbb{R}^d \mid \hat{q} = \sum_{i=0}^{d} \tilde{\lambda}_i(\hat{q})\hat{q}^i, \text{ with } \tilde{\lambda}_i(\hat{q}) \geq 0, \text{ and } \sum_{i=0}^{d} \tilde{\lambda}_i(\hat{q}) = 1 \right\}. \tag{5}
\]
As immediate property, the barycentric coordinates \( (\tilde{\lambda}_i)_{i=0}^{d} \) on \( \tilde{K}^d \) satisfy
\[
\tilde{\lambda}_i(\hat{q}') = \delta_{i,j}, \quad \forall (i, j) \in [0, d]. \tag{6}
\]
and are explicitly given with \( \hat{q} = (x_1, \ldots, x_d)^T \in \tilde{K}^d \) by
\[
\tilde{\lambda}_0(\hat{q}) = 1 - \sum_{i=1}^{d} x_i \quad \text{and } \forall i \in [1, d], \quad \tilde{\lambda}_i(\hat{q}) = x_i. \tag{7}
\]
Indeed, as \( \hat{q}^0 = 0 \), we have
\[
\hat{q} = \sum_{i=0}^{d} \tilde{\lambda}_i(\hat{q})\hat{q}^i = \sum_{i=1}^{d} \hat{q}^i \tilde{\lambda}_i(\hat{q}).
\]
From \( \hat{q}^i = \hat{e}^i, \ \forall i \in [1, d] \), we obtain
\[
\sum_{i=1}^{d} \hat{q}^i \tilde{\lambda}_i(\hat{q}) = \begin{pmatrix} \hat{q}^1 & \cdots & \hat{q}^d \end{pmatrix} \begin{pmatrix} \tilde{\lambda}_1(\hat{q}) \\ \vdots \\ \tilde{\lambda}_d(\hat{q}) \end{pmatrix} = I_d \begin{pmatrix} \tilde{\lambda}_1(\hat{q}) \\ \vdots \\ \tilde{\lambda}_d(\hat{q}) \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}_1(\hat{q}) \\ \vdots \\ \tilde{\lambda}_d(\hat{q}) \end{pmatrix}
\]
and thus
\[
\hat{q} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}_1(\hat{q}) \\ \vdots \\ \tilde{\lambda}_d(\hat{q}) \end{pmatrix}.
\]
From (5), we have
\[
\sum_{i=0}^{d} \tilde{\lambda}_i(\hat{q}) = 1
\]
and thus
\[
\tilde{\lambda}_0(\hat{q}) = 1 - \sum_{i=1}^{d} \tilde{\lambda}_i(\hat{q}) = 1 - \sum_{i=1}^{d} x_i.
\]

2.1 unit d-simplex volume

There are several ways to compute the volume \( |\hat{K}| \) of the d-simplex \( \hat{K} \subset \mathbb{R}^d \) which is given by the following integral:
\[
|\hat{K}| = \int_{\hat{K}} 1 \, d\hat{q} = \int_{0}^{1} \int_{0}^{1-x_1} \int_{0}^{1-x_1-x_2} \cdots \int_{0}^{1-(x_1+\ldots+x_{d-1})} 1 \, dx_d \ldots dx_3 dx_2 dx_1.
\]
An elegant way to perform this integration is explained in [5], section 18.10, and uses a Laplace transform. To use this method, we note that

\[ \hat{K} = \mathbb{R}^d_+ \cap \{1 - (x_1 + \ldots + x_d) \geq 0\}. \]  

(8)

So we also have

\[ |\hat{K}| = \int_{\mathbb{R}^d_+ \cap \{1-(x_1+\ldots+x_d)\geq 0\}} 1 \, dx_d \ldots dx_1. \]

By using a Dirac function and extending the integration domain to \( \mathbb{R}^{d+1}_+ \), we also have

\[ |\hat{K}| = \int_{\mathbb{R}^{d+1}_+} \delta(x_1 + \ldots + x_d + x_{d+1} - 1) \, dx_{d+1} \, dx_d \ldots dx_1. \]

To use the Laplace transform theory, we define the function \( f \) by

\[ f(t) = \int_{\mathbb{R}^{d+1}_+} \delta(x_1 + \ldots + x_d + x_{d+1} - t) \, dx_{d+1} \, dx_d \ldots dx_1 \]

so that \( |\hat{K}| = f(1) \). The Laplace transform of \( f \) is given by

\[
\mathcal{L}(f)(s) = \int_0^{\infty} f(t)e^{-st} \, dt \\
= \int_{\mathbb{R}^{d+1}_+} \left( \int_0^{\infty} \delta(x_1 + \ldots + x_d + x_{d+1} - t) e^{-st} \, dt \right) \, dx_{d+1} \, dx_d \ldots dx_1 \\
= \int_{\mathbb{R}^{d+1}_+} \exp(-s \sum_{i=1}^{d+1} x_i) \, dx_{d+1} \, dx_d \ldots dx_1 \\
= \prod_{i=1}^{d+1} \int_0^{\infty} \exp(-sx_i) \, dx_i \\
= \frac{1}{s^{d+1}}.
\]

By using the inverse Laplace transform table (see [5] for example), we have

\[
\mathcal{L}^{-1}(s \mapsto \frac{d!}{s^{d+1}})(t) = t^d.
\]

As \( f = \mathcal{L}^{-1} \circ \mathcal{L}(f) \) and by linearity of the inverse Laplace transform we obtain

\[ f(t) = \frac{t^d}{d!}. \]

So the volume of the unit \( d \)-simplex is

\[ |\hat{K}| = \frac{1}{d!}. \]  

(9)
2.2 Magic formula

Let \( \nu = (\nu_0, \ldots, \nu_d) \in \mathbb{N}^{d+1} \). The magic formula is given by

\[
\int_{\mathbb{R}} \prod_{i=0}^{d} \hat{\lambda}_i(q) dq = \frac{\prod_{i=0}^{d} \nu_i!}{(d + \sum_{i=0}^{d} \nu_i)!}
\] (10)

This formula is often used in \( P^1 \)-Lagrange finite element methods because \( P^1 \)-Lagrange basis functions on a d-simplex are the associated barycentric coordinates. For example, one can refer to [7] (section 8.2.1, page 179, formula (8.3)), [2], [4] section 7.3.3 page 126, [3] for \( d \in \{1, 3\} \), [2] as exercise for \( d = 2 \) and \( d = 3 \). In this section, we propose a proof of this formula using Laplace transform theory. Let \( \hat{I}(\nu) \) denote the integral of (10). The barycentric coordinates \( \hat{\lambda}_i \) are given in (7) and so with \( q = (x_1, \ldots, x_d) \) and using (8) we obtain

\[
\hat{I}(\nu) = \int_{\mathbb{R}} \left(1 - \sum_{i=1}^{d} x_i\right)^{\nu_0} \prod_{i=1}^{d} x_i^{\nu_i} dx_1 \ldots dx_1
\]

From section 2.2, by using a dirac function and by extending the integration domain to \( \mathbb{R}_{\geq0}^{d+1} \) we obtain with \( \nu_{d+1} = \nu_0 \)

\[
\hat{I}(\nu) = \int_{\mathbb{R}_{\geq0}^{d+1}} \delta(x_1 + \ldots + x_d + x_{d+1} - 1) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} \ldots dx_1
\]

To use the Laplace transform theory, we define the function \( f_\nu \) by

\[
f_\nu(t) = \int_{\mathbb{R}_{\geq0}^{d+1}} \delta(x_1 + \ldots + x_d + x_{d+1} - t) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} \ldots dx_1
\]

so that \( \hat{I}(\nu) = f_\nu(1) \). The Laplace transform of \( f_\nu \) is given by

\[
\mathcal{L}(f_\nu)(s) = \int_0^\infty f_\nu(t) e^{-st} dt
\]

\[
= \int_{\mathbb{R}_{\geq0}^{d+1}} \left( \int_0^\infty \delta(x_1 + \ldots + x_d + x_{d+1} - t) e^{-st} dt \right) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} \ldots dx_1
\]

\[
= \int_{\mathbb{R}_{\geq0}^{d+1}} \exp(-s \sum_{i=1}^{d+1} x_i) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} \ldots dx_1
\]

\[
= \prod_{i=1}^{d+1} \int_0^\infty x_i^{\nu_i} \exp(-sx_i) dx_i
\]

\[
= \prod_{i=1}^{d+1} \mathcal{L}(t \mapsto t^{\nu_i})(s)
\]
In a classical Laplace transform table (see [8] for example), we have

\[ \mathcal{L}(t \mapsto t^k)(s) = \frac{1}{s^{k+1}} \]

and by linearity of the Laplace transform

\[ \mathcal{L}(t \mapsto t^k)(s) = \frac{k!}{s^{k+1}}. \]

So we obtain

\[ \mathcal{L}(f\nu)(s) = \prod_{i=1}^{d+1} \frac{\nu_i!}{\nu_{i+1}!} = \frac{\prod_{i=1}^{d+1} \nu_i!}{s^{d+1+\sum_{i=1}^{d+1} \nu_i}} \]

By using the inverse Laplace transform table, we have

\[ \mathcal{L}^{-1}(s \mapsto \frac{1}{s^k})(t) = \frac{t^{k-1}}{k-1}. \]

With the linearity of the inverse Laplace transform we obtain

\[ f\nu(t) = \mathcal{L}^{-1}(\mathcal{L}(f\nu)(s))(t) = \prod_{i=1}^{d+1} \frac{\nu_i!}{\nu_{i+1}!} t^{d+\sum_{i=1}^{d+1} \nu_i}. \]

As \( \hat{f}(\nu) = f\nu(1) \) and \( \nu_{d+1} = \nu_0 \), the equation (10) is proved.

3 Some results on a d-simplex in \( \mathbb{R}^n \)

3.1 Gradients of Barycentric coordinates on a d-simplex

Lemma 3 Let \( K \subset \mathbb{R}^n \) be a non-degenerate d-simplex and and \( \{ \mathbf{q}^i \}_{i=0}^{d} \) its vertices. The barycentric coordinates \( (\lambda_i(\mathbf{q}))_{i=0}^{d} \) are solution of the linear system

\[ \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & \ddots & \ddots & 0 \\ 0 & \cdots & \ddots & \ddots \\ \mathbb{A}_K \mathbb{A}_K \end{bmatrix} \begin{bmatrix} \lambda_0(\mathbf{q}) \\ \lambda_1(\mathbf{q}) \\ \vdots \\ \lambda_d(\mathbf{q}) \end{bmatrix} = \begin{bmatrix} 1 \\ \mathbb{A}_K(\mathbf{q} - \mathbf{q}^0) \end{bmatrix} \] (11)

where \( \mathbb{A}_K \in \mathcal{M}_{n,d}(\mathbb{R}) \) is defined by

\[ \mathbb{A}_K = \begin{bmatrix} \mathbf{q}^1 - \mathbf{q}^0 & \cdots & \mathbf{q}^d - \mathbf{q}^0 \end{bmatrix} \] (12)

The barycentric coordinates are multivariate polynomials of first degree and their gradients are given by

\[ ( \nabla \lambda_1(\mathbf{q}) \; \cdots \; \nabla \lambda_d(\mathbf{q}) ) = \mathbb{A}_K (\mathbb{A}_K^t \mathbb{A}_K)^{-t} \] (13)

and

\[ \nabla \lambda_0(\mathbf{q}) = -\sum_{i=1}^{d} \nabla \lambda_i(\mathbf{q}). \] (14)
Proof: As \( \sum_{i=0}^{d} \lambda_i(q) = 1 \), we have

\[
q = \sum_{i=0}^{d} \lambda_i(q) q^i = q - q^0 = \sum_{i=1}^{d} (q^i - q^0) \lambda_i(q) = \delta_K \left( \begin{array}{c} \lambda_1(q) \\ \vdots \\ \lambda_d(q) \end{array} \right)
\]

Due to linear independence of \( (q^i - q^0)_{i=1}^{d} \),

\[
\mathcal{H}_K \overset{\text{def}}{=} \delta_K^T \delta_K \in \mathcal{M}_{d,d}(\mathbb{R})
\]

is a regular matrix and the barycentric coordinates are solution of the linear system

\[
\delta_K^T \delta_K \left( \begin{array}{c} \lambda_1(q) \\ \vdots \\ \lambda_d(q) \end{array} \right) = \delta_K^T (q - q^0) \quad \text{and} \quad \sum_{i=0}^{d} \lambda_i(q) = 1.
\]

In matrix form these equations can be written as \([11]\) and we deduce that the barycentric coordinates \( \lambda_i \) are multivariate polynomials of first degree. So their gradients are constants on \( K \).

The affine map/transformation \( \mathcal{F}_K \) from the unit \( d \)-simplex \( \hat{K} \subset \mathbb{R}^d \) to \( K \subset \mathbb{R}^n \) is given by

\[
q = \delta_K \hat{q} + q^0 = \mathcal{F}_K(\hat{q}).
\]

So we have

\[
\delta_K^T (q - q^0) = \delta_K^T \delta_K \hat{q} = \mathcal{H}_K \hat{q}
\]

and thus \( \mathcal{F}_K^{-1} : K \subset \mathbb{R}^n \rightarrow \hat{K} \subset \mathbb{R}^d \) is defined by

\[
\hat{q} = \mathcal{H}_K^{-1} \delta_K (q - q^0) = \mathcal{F}_K^{-1}(q).
\]

So we have

\[
\lambda_i(q) = (\hat{\lambda}_i \circ \mathcal{F}_K^{-1})(q) \quad \text{and} \quad \hat{\lambda}_i(q) = (\lambda_i \circ \mathcal{F}_K)(\hat{q})
\]

One can remark that if \( d = n \) then \( \delta_K \) is a regular square matrix and \( \mathcal{H}_K \overset{\text{def}}{=} \delta_K^T \delta_K = \delta_K^2 \).

Now we may compute partial derivative of \( \lambda_i \) and \( \forall i \in [0, d], \forall j \in \{1, n\} \), we obtain with \( \hat{q} = (\hat{x}_1, \ldots, \hat{x}_d) \) and \( q = (x_1, \ldots, x_n) \)

\[
\frac{\partial \lambda_i}{\partial x_j}(q) = \sum_{l=1}^{d} \frac{\partial \hat{\lambda}_i}{\partial \hat{x}_j}(\mathcal{F}_K^{-1}(q)) \frac{\partial \mathcal{F}_K^{-1}(q)}{\partial x_j}(q)
\]

From \([17]\) denoting \( \mathcal{B}_K = \mathcal{H}_K^{-1} \delta_K \in \mathcal{M}_{d,m}(\mathbb{R}) \) gives \( \frac{\partial \mathcal{F}_K^{-1}}{\partial x_j}(q) = (\mathcal{B}_K)_{i,j} \). The barycentric coordinates are polynomials of first degree, so their gradients are constants and we obtain

\[
\nabla \lambda_i = \mathcal{B}_K^T \nabla \hat{\lambda}_i
\]

(in fact \( \mathcal{B}_K \) is the Jacobian matrix of \( \mathcal{F}_K^{-1} \)). The matrix \( \mathcal{H}_K \) is regular and symmetric, so \( \mathcal{B}_K = \delta_K \mathcal{H}_K^{-1} \) and we obtain

\[
\nabla \lambda_i = \delta_K \mathcal{H}_K^{-1} \nabla \hat{\lambda}_i.
\]

(19)
From (7), we deduced that
\[ \left( \nabla \hat{\lambda}_1 \; \cdots \; \nabla \hat{\lambda}_d \right) = 0_d \] (20)
and thus
\[ \left( \nabla \lambda_1(q) \; \cdots \; \nabla \lambda_d(q) \right) = A_K \hat{H}_K^{-1} \left( \nabla \hat{\lambda}_1 \; \cdots \; \nabla \hat{\lambda}_d \right) = A_K \hat{H}_K^{-1}. \]
As \( \sum_{i=0}^d \lambda_i(q) = 1 \), we immediately have
\[ \nabla \lambda_0(q) = - \sum_{i=1}^d \nabla \lambda_i(q). \]

From (13) and (14), we immediately have:

**Remark 4** The gradients of the barycentric coordinates are linear combinations of \( q_1 - q_0, \ldots, q_d - q_0 \).

### 3.2 Integration over a \( d \)-simplex

If \( K \) is a non-degenerated \( d \)-simplex in \( \mathbb{R}^d \), from (16) we have \( J_{\mathcal{F}_K}(\hat{q}) = A_K \). Then \( A_K \) is a regular square matrix and we have the classical formula:
\[ \int_K f(q) dq = |\text{det}(A_K)| \int_{K} f \circ \mathcal{F}_K(\hat{q}) d\hat{q} \] (21)
The following theorem extend this result to \( d \)-simplex in \( \mathbb{R}^n \), with \( 1 \leq d \leq n \).

**Theorem 5** Let \( K \subset \mathbb{R}^n \) be a non-degenerated \( d \)-simplex and \( f : K \rightarrow \mathbb{R} \).
\[ \int_K f(q) dq = |\text{det}(A_K A_K^T)|^{1/2} \int_K f \circ \mathcal{F}_K(\hat{q}) d\hat{q} \] (22)
where \( K \) is the unit \( d \)-simplex in \( \mathbb{R}^n \), \( A_K \in \mathcal{M}_{d,n}(\mathbb{R}) \) is defined by
\[ A_K = \left( \begin{array}{c} q_1 - q_0 \; q_2 - q_0 \; \cdots \; q_d - q_0 \end{array} \right) \] (23)
and \( \mathcal{F}_K : K \rightarrow K \) is given by
\[ \mathcal{F}_K(\hat{q}) = A_K \hat{q} + q_0 \] (24)

**Proof:** The set \( \{v^1, \ldots, v^d\} \) is linearly independent so we can extend it to a basis \( \{v^1, \ldots, v^n\} \). We denote by \( A \in \mathcal{M}_{n,n}(\mathbb{R}) \) the matrix such that the \( i \)-th column is the vector \( v^i \) for all \( i \in [1, n] \). So we have
\[ A = \left( \begin{array}{c|c|c} A_K & v^{d+1} & \cdots & v^n \end{array} \right) \] (25)
By the QR-factorization theorem applied to the matrix \( A \in \mathcal{M}_{n,n}(\mathbb{R}) \), there is an orthogonal matrix \( Q \in \mathcal{M}_{n,n}(\mathbb{R}) \) and a regular upper triangular matrix \( R \in \mathcal{M}_{n,n}(\mathbb{R}) \) such that
\[ A = QR \]
\[ Q^L A = \mathbb{R} \]
and we define the matrix \( \hat{A} \in \mathcal{M}_{n,d}(\mathbb{R}) \) to be the first \( d \) columns of \( \mathbb{R}^d \):
\[ \hat{A} = Q^L A_K. \]

We can also note that
\[
\hat{A} = \begin{pmatrix}
    \mathbf{q}^1 - \mathbf{q}^0 \\
    \mathbf{q}^2 - \mathbf{q}^0 \\
    \vdots \\
    \mathbf{q}^d - \mathbf{q}^0
\end{pmatrix}
= \begin{pmatrix}
    \mathbf{q}^1 \\
    \mathbf{q}^2 \\
    \vdots \\
    \mathbf{q}^d
\end{pmatrix}.
\]

Let \( \mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the bijective function defined by
\[
\mathcal{F}(x) = Q^L (x - \mathbf{q}^0) = \hat{x}
\]
and
\[
\mathbf{q}^i = \mathcal{F}^{-1}(\mathbf{q}^i) = Q^L (\mathbf{q}^i - \mathbf{q}^0), \quad \forall i \in [0,d].
\]
By construction \( \mathbf{q}^0 = 0 \) and, \( \forall i \in [1,d], \mathbf{q}^i \) is the \( i \)-th column of the upper triangular matrix \( \mathbb{R} \). So we have
\[
\forall i \in [0,d], \quad \mathbf{q}^i \in \text{Vect}(e^1, \ldots, e^d)
\]
where \( \{e^1, \ldots, e^n\} \) is the standard basis of \( \mathbb{R}^n \). The set \( \{\mathbf{q}^0, \ldots, \mathbf{q}^d\} \) are the vertices of the \( d \)-simplex \( \mathcal{K} = \mathcal{F}(K) \) and we deduce
\[
\mathcal{K} \subset \text{Vect}(e^1, \ldots, e^d).
\]

By change of variables, we obtain
\[
\int_K f(\mathbf{q}) d\mathbf{q} = \int_{\mathcal{K}} f \circ \mathcal{F}^{-1}(\mathbf{q}) \det(J_{\mathcal{F}^{-1}}(\mathbf{q})) |d\mathbf{q}|
\]
where \( J_{\mathcal{F}^{-1}} \) is the Jacobian matrix of \( \mathcal{F}^{-1} \). From [26], we have \( J_{\mathcal{F}^{-1}}(\mathbf{q}) = Q \) and as \( Q \) is an orthogonal matrix, \( \det(J_{\mathcal{F}^{-1}}(\mathbf{q})) = 1 \). So we obtain
\[
\int_K f(\mathbf{q}) d\mathbf{q} = \int_{\mathcal{K}} f \circ \mathcal{F}^{-1}(\mathbf{q}) |d\mathbf{q}|.
\]

Let \( P \in \mathcal{M}_{d,n}(\mathbb{R}) \) defined by
\[
P = \left( \begin{array}{c|c}
    I_d & 0_{d,n-d}
\end{array} \right)
\]
and
\[
\forall i \in [0,d], \quad \mathbf{q}^i = P \mathbf{q}^i \in \mathbb{R}^d.
\]

From [27], we deduce
\[
\forall i \in [0,d], \quad \mathbf{q}^i = P \mathbf{q}^i = \begin{pmatrix}
    \mathbf{q}^i \\
    0
\end{pmatrix}.
\]

Let \( \tilde{g} = f \circ \mathcal{F}^{-1} \) and \( \tilde{K} \) be the \( d \)-simplex in \( \mathbb{R}^d \) with vertices \( \mathbf{q}^i, i \in [0,d] \). We denote by \( P : \tilde{K} \subset \mathbb{R}^d \rightarrow K \subset \mathbb{R}^n \) the application defined by \( P(\mathbf{q}) = P\mathbf{q} \). We denote by \( \tilde{g} : \tilde{K} \rightarrow \mathbb{R} \) the application defined by
\[
\tilde{g}(\mathbf{q}) = \tilde{g} \circ P(\mathbf{q}) = \tilde{g} \begin{pmatrix}
    \mathbf{q} \\
    0
\end{pmatrix}.
\]
So we obtain

$$\int_{\bar{K}} \bar{g}(\bar{q}) d\bar{q} = \int_{\bar{K}} \bar{g}(\bar{q}) d\bar{q}$$  \hspace{1cm} (29)$$

Let $\bar{A} \in \mathcal{M}_d(\mathbb{R})$ be the matrix defined by

$$\bar{A} = \begin{pmatrix} \bar{q}^1 - \bar{q}^0 & \bar{q}^2 - \bar{q}^0 & \cdots & \bar{q}^d - \bar{q}^0 \end{pmatrix}. \hspace{1cm} (30)$$

We can remark that

$$\bar{A} = P\bar{A} \hspace{1cm} \text{and} \hspace{1cm} \bar{A} = \bar{P}^T \bar{A}.$$  

Let $\bar{F} : \bar{K} \longrightarrow \bar{K}$ the bijective function defined by

$$\bar{F}(\bar{q}) = \bar{F} + \bar{q}^0$$

We can now apply the classical change of variables

$$\int_{\bar{K}} \bar{g}(\bar{q}) d\bar{q} = \int_{\bar{K}} \bar{g} \circ \bar{F}(\bar{q}) |\det(\bar{F}(\bar{q}))| d\bar{q}$$

$$= \left| \det(\bar{A}) \right| \int_{\bar{K}} \bar{g} \circ \bar{F}(\bar{q}) d\bar{q}$$

To resume from (22) and (29), we have

$$\int_{\bar{K}} f(\bar{q}) d\bar{q} = |\det(\bar{A})| \int_{\bar{K}} \bar{g} \circ \bar{F}(\bar{q}) d\bar{q} \hspace{1cm} (31)$$

We can note that

$$\bar{g} \circ \bar{F} = f \circ \bar{F}^{-1} \circ \bar{P} \circ \bar{F}$$

Let $\bar{F}_K = \bar{F}^{-1} \circ \bar{P} \circ \bar{F}$, we have as expected

$$\bar{F}_K(\bar{q}) = \bar{F}^{-1} \circ \bar{P} \circ \bar{F}(\bar{q})$$

$$= \bar{F}^{-1}(\bar{P}(\bar{q}))$$

$$= \bar{F}^{-1}(\bar{\bar{q}})$$

$$= Q\bar{\bar{q}} + \bar{q}^0$$

$$= A_K \bar{q} + \bar{q}^0.$$  

and we obtain

$$\int_{\bar{K}} f(\bar{q}) d\bar{q} = |\det(\bar{A})| \int_{\bar{K}} f \circ \bar{F}_K(\bar{q}) d\bar{q} \hspace{1cm} (32)$$

To obtain formula (22), it remains to prove that $|\det(\bar{A})| = |\det(A_K^t A_K)|^{1/2}$. We have

$$A_K^t A_K = A_K^t Q Q^t A_K \hspace{1cm} \text{as} \hspace{1cm} A_K = Q\bar{\bar{A}}$$

$$= A^t \bar{\bar{A}} \hspace{1cm} \text{as} \hspace{1cm} \bar{Q} \text{ is an orthogonal matrix}$$

$$= A^t P P^t \bar{\bar{A}} \hspace{1cm} \text{as} \hspace{1cm} \bar{A} = P \bar{\bar{A}}$$

$$= \bar{A}^t \bar{A} \hspace{1cm} \text{as} \hspace{1cm} PP^t = I_d$$

As $\bar{A}$ is a square matrix, we have $\det(\bar{A}^t \bar{A}) = \det(\bar{A})^2$ and thus

$$|\det(\bar{A})| = |\det(A_K^t A_K)|^{1/2}.$$  

\[\square\]
3.3 Volume of a d-simplex

The volume of the d-simplex $K \subset \mathbb{R}^n$ is given by

$$|K| = \int_{K} 1 d\mathbf{q}$$

(33)

Using formula (22) with $f = 1$ gives

$$|K| = |\det(\mathbf{A}_K^x \mathbf{A}_K)|^{1/2} \int_{K} 1 d\mathbf{q} = |\det(\mathbf{A}_K^x \mathbf{A}_K)|^{1/2} |\hat{K}|.$$  

From (32), we finally obtain

$$|K| = \frac{|\det(\mathbf{A}_K^x \mathbf{A}_K)|^{1/2}}{d!}.$$  

(34)

In [3] this formula is proved with geometrical arguments. We can also remark that if $d = n$ then $\mathbf{A}_K$ is a square matrix and we obtain the classical formula

$$|K| = \frac{|\det(\mathbf{A}_K)|}{d!}.$$  

(35)

3.4 Magic formula

In this section an exact computation of the integral over a d-simplex $K \subset \mathbb{R}^n$ for products of power of its barycentric coordinates given by (1) is proved by using previous results obtained by Laplace transforms.

Using formula (22) with $f(\mathbf{q}) = \prod_{i=0}^{d} \lambda_i^\nu(\mathbf{q})$ gives

$$\int_{K} \prod_{i=0}^{d} \lambda_i^\nu(\mathbf{q}) d\mathbf{q} = |\det(\mathbf{A}_K^x \mathbf{A}_K)|^{1/2} \int_{K} \prod_{i=0}^{d} (\lambda_i \circ F_K(\mathbf{q}))^\nu d\mathbf{q}$$

From (18) and (34), we obtain

$$\int_{K} \prod_{i=0}^{d} \lambda_i^\nu(\mathbf{q}) d\mathbf{q} = d! |K| \int_{K} \prod_{i=0}^{d} \hat{\lambda}_i^\nu(\mathbf{q}) d\mathbf{q}$$

Using formula (10) gives (1).

References


[9] F.J. Vermolen and A. Segal. On an integration rule for products of barycen-
tric coordinates over simplexes in rn. *Journal of Computational and Applied