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Exact integration for products of power of barycentric coordinates over d-simplexes in $\mathbb{R}^n$

François Cuvelier

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Abstract

Exact integral computation over a d-simplex in $\mathbb{R}^n$ for products of powers of its barycentric coordinates is done in [9] by using mathematical induction and coordinate mappings. In this note we give a new proof using Laplace transformations without mathematical induction.

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∗Université Paris 13, Sorbonne Paris Cité, LAGA, CNRS UMR 7539, 99 Avenue J-B Clément, F-93430 Villetaneuse, France, cuvelier@math.univ-paris13.fr

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Local shape functions of a large variety of finite element on a d-simplex $K \subset \mathbb{R}^n$ can be expressed in function of the barycentric coordinates $\{\lambda_0, \ldots, \lambda_d\}$ of $K$ and their derivatives (see [1] for examples).

In [9], the authors give a proof of the magic formula: let $\nu = (\nu_0, \ldots, \nu_d) \in \mathbb{N}^{d+1}$, then

$$
\int_K \prod_{i=0}^{d} \lambda_i^{\nu_i}(q) dq = d! |K| \frac{\prod_{i=0}^{d} \nu_i!}{(d + \sum_{i=0}^{d} \nu_i)!}
$$

where $|K|$ is the volume of $K$. In their proof, mathematical induction and coordinate mappings are mainly used. In this note we give a new proof of this formula using Laplace transformations without mathematical induction.

Firstly we recall definitions of a d-simplex in $\mathbb{R}^n$ and of its barycentric coordinates. Then we introduce Laplace transforms to compute the volume of the unit d-simplex $\hat{K} \subset \mathbb{R}^d$ and the magic formula (1) over $\hat{K}$. In the last section, we propose to compute the gradients of the barycentric coordinates by solving linear systems. We also present the mapping of an integral over a d-simplex in $\mathbb{R}^n$ to the reference unit d-simplex, allowing to prove (1).

1 Notations and definitions

Let $n \in \mathbb{N}^*$ be the space dimension and $d \in [0, n]$. We recall the definition of a d-simplex in $\mathbb{R}^n$ as well as its barycentric coordinates.

**Definition 1 (d-simplex)** A d-simplex $K \subset \mathbb{R}^n$ is the convex hull of $(d + 1)$ points $q^0, \ldots, q^d$ of $\mathbb{R}^n$ which form the vertices of $K$.

$$K = \left\{ q \in \mathbb{R}^n \mid q = \sum_{i=0}^{d} \theta_i q^i, \text{ with } \forall i \in [0, d], \theta_i \geq 0, \text{ and } \sum_{i=0}^{d} \theta_i = 1 \right\}. \quad (2)$$

For example, a 2-simplex is a triangle and a 3-simplex is a tetrahedron. It will be always assumed that a d-simplex is not degenerated, i.e., the set of vectors $\{q^i - q^0\}_{i=1}^{d}$ is linearly independent.

**Definition 2 (Barycentric coordinates)** Let $K \subset \mathbb{R}^n$ be a non-degenerate d-simplex and $\{q^i\}_{i=0}^{d}$ its vertices. The parametrization of $K$ with a convex combination of the vertices reads as follows

$$K = \left\{ q \in \mathbb{R}^n \mid q = \sum_{i=0}^{d} \lambda_i q^i, \text{ with } \forall i \in [0, d], \lambda_i(q) \geq 0, \text{ and } \sum_{i=0}^{d} \lambda_i(q) = 1 \right\}. \quad (3)$$

The coefficients $\lambda_0(q), \ldots, \lambda_d(q)$ are called the barycentric coordinates on $K$ of $q$.

As immediate property, the barycentric coordinates on $K$ satisfy

$$\lambda_i(q^j) = \delta_{i,j}, \forall (i, j) \in [0, d]. \quad (4)$$
2 Some results on the unit d-simplex

The unit d-simplex $\hat{K}^d \subset \mathbb{R}^d$ is defined by the $d+1$ vertices

$$\{\hat{q}, \hat{q}^1, \ldots, \hat{q}^d\} = \{0, \hat{e}^1, \ldots, \hat{e}^d\}$$

where $\{\hat{e}^1, \ldots, \hat{e}^d\}$ is the standard basis of $\mathbb{R}^d$. We have

$$\hat{K}^d = \left\{ \hat{q} \in \mathbb{R}^d \mid \hat{q} = \sum_{i=0}^{d} \lambda_i(\hat{q}) \hat{q}^i, \text{ with } \lambda_i(\hat{q}) \geq 0, \text{ and } \sum_{i=0}^{d} \lambda_i(\hat{q}) = 1 \right\}. \quad (5)$$

As immediate property, the barycentric coordinates $\lambda_i(\hat{q})$ on $\hat{K}^d$ satisfy

$$\lambda_i(\hat{q}^j) = \delta_{i,j}, \quad \forall (i, j) \in [0, d]. \quad (6)$$

and are explicitly given with $\hat{q} = (x_1, \ldots, x_d)^t \in \hat{K}^d$ by

$$\lambda_0(\hat{q}) = 1 - \sum_{i=1}^{d} x_i \quad \text{and} \quad \lambda_i(\hat{q}) = x_i. \quad (7)$$

Indeed, as $\hat{q}^0 = 0$, we have

$$\hat{q} = \sum_{i=0}^{d} \lambda_i(\hat{q}) \hat{q}^i = \sum_{i=1}^{d} \hat{q}^i \lambda_i(\hat{q}).$$

From $\hat{q}^i = \hat{e}^i$, $\forall i \in [1, d]$, we obtain

$$\sum_{i=1}^{d} \hat{q}^i \lambda_i(\hat{q}) = \left( \hat{q}^1 \ldots \hat{q}^d \right) \begin{pmatrix} \lambda_1(\hat{q}) \\ \vdots \\ \lambda_d(\hat{q}) \end{pmatrix} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \begin{pmatrix} \lambda_1(\hat{q}) \\ \vdots \\ \lambda_d(\hat{q}) \end{pmatrix} = \begin{pmatrix} \lambda_1(\hat{q}) \\ \vdots \\ \lambda_d(\hat{q}) \end{pmatrix}$$

and thus

$$\hat{q} = \begin{pmatrix} x_1 \\ \vdots \\ x_d \end{pmatrix} = \begin{pmatrix} \lambda_1(\hat{q}) \\ \vdots \\ \lambda_d(\hat{q}) \end{pmatrix}.$$

From $\hat{q}^0$, we have

$$\sum_{i=0}^{d} \lambda_i(\hat{q}) = 1$$

and thus

$$\lambda_0(\hat{q}) = 1 - \sum_{i=1}^{d} \lambda_i(\hat{q}) = 1 - \sum_{i=1}^{d} x_i.$$  

2.1 unit d-simplex volume

There are several ways to compute the volume $|\hat{K}|$ of the d-simplex $\hat{K} \subset \mathbb{R}^d$ which is given by the following integral:

$$|\hat{K}| = \int_{\hat{K}} 1 d\hat{q} = \int_{0}^{1} \int_{0}^{1-x_1} \int_{0}^{1-x_1-x_2} \cdots \int_{0}^{1-(x_1+\ldots+x_{d-1})} 1 dx_d \ldots dx_2 dx_1.$$
An elegant way to perform this integration is explained in [5], section 18.10, and uses a Laplace transform. To use this method, we note that

\[
\hat{K} = \mathbb{R}^d_+ \cap \{1 - (x_1 + \ldots + x_d) \geq 0\}.
\]

(8)

So we also have

\[
|\hat{K}| = \int_{\mathbb{R}^d_+ \cap \{1 - (x_1 + \ldots + x_d) \geq 0\}} 1 \, dx_d \ldots dx_1.
\]

By using a dirac function and extending the integration domain to \(\mathbb{R}^{d+1}_+\), we also have

\[
|\hat{K}| = \int_{\mathbb{R}^{d+1}_+} \delta(x_1 + \ldots + x_d + x_{d+1} - 1) \, dx_{d+1} dx_d \ldots dx_1.
\]

To use the Laplace transform theory, we define the function \(f\) by

\[
f(t) = \int_{\mathbb{R}^{d+1}_+} \delta(x_1 + \ldots + x_d + x_{d+1} - t) \, dx_{d+1} dx_d \ldots dx_1
\]

so that \(\hat{K} = f(1)\). The Laplace transform of \(f\) is given by

\[
\mathcal{L}(f)(s) = \int_0^\infty f(t) e^{-st} \, dt
\]

\[
= \int_{\mathbb{R}^{d+1}_+} \left( \int_0^\infty \delta(x_1 + \ldots + x_d + x_{d+1} - t) e^{-st} \, dt \right) \, dx_{d+1} dx_d \ldots dx_1
\]

\[
= \int_{\mathbb{R}^{d+1}_+} \exp(-s \sum_{i=1}^{d+1} x_i) dx_{d+1} dx_d \ldots dx_1
\]

\[
= \prod_{i=1}^{d+1} \int_0^\infty \exp(-sx_i) \, dx_i
\]

\[
= \frac{1}{s^{d+1}}.
\]

By using the inverse Laplace transform table (see [8] for example), we have

\[
\mathcal{L}^{-1}(s \mapsto \frac{d!}{s^{d+1}})(t) = t^d.
\]

As \(f = \mathcal{L}^{-1} \circ \mathcal{L}(f)\) and by linearity of the inverse Laplace transform we obtain

\[
f(t) = \frac{t^d}{d!}
\]

So the volume of the unit \(d\)-simplex is

\[
|\hat{K}| = \frac{1}{d!}
\]

(9)
2.2 Magic formula

Let $\nu = (\nu_0, \ldots, \nu_d) \in \mathbb{N}^{d+1}$. The magic formula is given by

$$
\int_R \prod_{i=0}^d \lambda^{\nu_i}(q_i) dq_i = \frac{\prod_{i=0}^d \nu_i!}{(d + \sum_{i=0}^d \nu_i)!}
$$

(10)

This formula is often used in $P^1$-Lagrange finite element methods because $P^1$-Lagrange basis functions on a $d$-simplex are the associated barycentric coordinates. For example, one can refer to [7] (section 8.2.1, page 179, formula (8.3)), [2], [3] section 7.3.3 page 126, [3] for $d \in \{1, 3\}$, [2] as exercise for $d = 2$ and $d = 3$. In this section, we propose a proof of this formula using Laplace transform theory. Let $\hat{I}(\nu)$ denote the integral of (10). The barycentric coordinates $\hat{\lambda}_i$ are given in (7) and so with $\hat{q} = (x_1, \ldots, x_d)$ and using [3] we obtain

$$
\hat{I}(\nu) = \int_R (1 - \sum_{i=1}^d x_i)^{\nu_0} \prod_{i=1}^d x_i^{\nu_i} dx_d \ldots dx_1
$$

$$
= \int_{R_{d+1}^d \cap \{1 - (x_1 + \ldots + x_d) > 0\}} (1 - \sum_{i=1}^d x_i)^{\nu_0} \prod_{i=1}^d x_i^{\nu_i} dx_d \ldots dx_1
$$

From section 2.1, by using a dirac function and by extending the integration domain to $R_{d+1}^{d+1}$ we obtain with $\nu_{d+1} = \nu_0$

$$
\hat{I}(\nu) = \int_{R_{d+1}^{d+1}} \delta(x_1 + \ldots + x_d + x_{d+1} - 1) x_{d+1}^{\nu_0} \prod_{i=1}^d x_i^{\nu_i} dx_{d+1} dx_d \ldots dx_1
$$

$$
= \int_{R_{d+1}^{d+1}} \delta(x_1 + \ldots + x_d + x_{d+1} - 1) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} dx_d \ldots dx_1
$$

To use the Laplace transform theory, we define the function $f_{\nu}$ by

$$
f_{\nu}(t) = \int_{R_{d+1}^{d+1}} \delta(x_1 + \ldots + x_d + x_{d+1} - t) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} dx_d \ldots dx_1
$$

so that $\hat{I}(\nu) = f_{\nu}(1)$. The Laplace transform of $f_{\nu}$ is given by

$$
\mathcal{L}(f_{\nu})(s) = \int_0^\infty f_{\nu}(t) e^{-st} dt
$$

$$
= \int_{R_{d+1}^{d+1}} \left( \int_0^\infty \delta(x_1 + \ldots + x_d + x_{d+1} - t) e^{-st} dt \right) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} dx_d \ldots dx_1
$$

$$
= \int_{R_{d+1}^{d+1}} \exp(-s \sum_{i=1}^{d+1} x_i) \prod_{i=1}^{d+1} x_i^{\nu_i} dx_{d+1} dx_d \ldots dx_1
$$

$$
= \prod_{i=1}^{d+1} \int_0^\infty x_i^{\nu_i} \exp(-sx_i) dx_i
$$

$$
= \prod_{i=1}^{d+1} \mathcal{L}(t \mapsto t^{\nu_i})(s)
$$
In a classical Laplace transform table (see [8] for example), we have
\[ \mathcal{L}(t \mapsto \frac{t^k}{k!}) (s) = \frac{1}{s^{k+1}} \]
and by linearity of the Laplace transform
\[ \mathcal{L}(t \mapsto t^k) (s) = \frac{k!}{s^{k+1}}. \]
So we obtain
\[ \mathcal{L}(f) (s) = \prod_{i=1}^{d+1} \frac{\nu_i^-1}{s^{\nu_i^-1}} = \prod_{i=1}^{d+1} \frac{\nu_i^-1}{s^{d+1+\sum_{i=1}^{i=1} \nu_i}}. \]
By using the inverse Laplace transform table, we have
\[ \mathcal{L}^{-1}(s \mapsto \frac{1}{s^k}) (t) = \frac{t^{k-1}}{k-1}. \]
With the linearity of the inverse Laplace transform we obtain
\[ f(t) = \mathcal{L}^{-1}(\mathcal{L}(f) (s)) (t) = \prod_{i=1}^{d+1} \frac{\nu_i^-1}{(d + \sum_{i=1}^{i=1} \nu_i)!} t^{d+\sum_{i=1}^{i=1} \nu_i} \]
As \( \hat{f}(v) = f(v)(1) \) and \( \nu_{d+1} = \nu_0 \), the equation (10) is proved.

3 Some results on a d-simplex in \( \mathbb{R}^n \)

3.1 Gradients of Barycentric coordinates on a d-simplex

Lemma 3 Let \( K \subset \mathbb{R}^n \) be a non-degenerate d-simplex and and \( \{ q_i \}^{d}_{i=0} \) its vertices. The barycentric coordinates \( (\lambda_i(q))^{d}_{i=0} \) are solution of the linear system
\[
\begin{pmatrix}
1 & 1 & \cdots & 1 \\
0 & 0 & \cdots & 0 \\
\mathcal{A}_K & \mathcal{A}_K & \cdots & \mathcal{A}_K \\
\end{pmatrix}
\begin{pmatrix}
\lambda_0(q) \\
\lambda_1(q) \\
\vdots \\
\lambda_d(q) \\
\end{pmatrix}
= \begin{pmatrix}
1 \\
0 \\
0 \\
0 \\
\end{pmatrix}
\]
where \( \mathcal{A}_K \in \mathcal{M}_{n,d}(\mathbb{R}) \) is defined by
\[ \mathcal{A}_K = \begin{pmatrix}
q^0 \\
q^1 \\
\vdots \\
q^d \\
\end{pmatrix}
\]
The barycentric coordinates are multivariate polynomials of first degree and their gradients are given by
\[
\nabla \lambda_i(q) = \left( \begin{array}{c} \nabla \lambda_1(q) \\
\vdots \\
\nabla \lambda_d(q) \\
\end{array} \right) = \mathcal{A}_K (\mathcal{A}_K^T \mathcal{A}_K)^{-1} \nabla \lambda_i(q).
\]
and
\[ \nabla \lambda_0(q) = -\sum_{i=1}^{d} \nabla \lambda_i(q). \]
Proof: As \( \sum_{i=0}^{d} \lambda_i(q) = 1 \), we have

\[
q = \sum_{i=0}^{d} \lambda_i(q)q^i \implies q - q^0 = \sum_{i=1}^{d} (q^i - q^0)\lambda_i(q) = \delta_K \begin{pmatrix} \lambda_1(q) \\ \vdots \\ \lambda_d(q) \end{pmatrix}
\]

Due to linear independence of \( \{q^i - q^0\}_{i=1}^{d} \),

\[
\mathcal{H}_K \overset{\text{def}}{=} \delta_K^T \delta_K \in \mathcal{M}_{d,d}(\mathbb{R})
\]

is a regular matrix and the barycentric coordinates are solution of the linear system

\[
\delta_K^T \delta_K \begin{pmatrix} \lambda_1(q) \\ \vdots \\ \lambda_d(q) \end{pmatrix} = \delta_K^T(q - q^0) \quad \text{and} \quad \sum_{i=0}^{d} \lambda_i(q) = 1.
\]

In matrix form these equations can be written as \([11]\) and we deduce that the barycentric coordinates \( \lambda_i \) are multivariate polynomials of first degree. So their gradients are constants on \( K \).

The affine map/transformation \( \mathcal{F}_K \) from the unit \( d \)-simplex \( \bar{K} \subset \mathbb{R}^d \) to \( K \subset \mathbb{R}^n \) is given by

\[
q = \delta_K \bar{q} + q^0 = \mathcal{F}_K(\bar{q}).
\]

So we have

\[
\delta_K^T(q - q^0) = \delta_K^T \delta_K \bar{q} = \mathcal{H}_K \bar{q}
\]

and thus \( \mathcal{F}_K^{-1} : K \subset \mathbb{R}^n \to \bar{K} \subset \mathbb{R}^d \) is defined by

\[
\bar{q} = \mathcal{H}_K^{-1} \delta_K(q - q^0) = \mathcal{F}_K^{-1}(q).
\]

So we have

\[
\lambda_i(q) = (\hat{\lambda}_i \circ \mathcal{F}_K^{-1})(q) \quad \text{and} \quad \hat{\lambda}_i(q) = (\lambda_i \circ \mathcal{F}_K)(\bar{q})
\]

One can remark that if \( d = n \) then \( \delta_K \) is a regular square matrix and \( \mathcal{H}_K^{-1} \delta_K^T = \delta_K^T \).

Now, we may compute partial derivative of \( \lambda_i \) and \( \forall i \in [0, d], \forall j \in [1, n] \), we obtain with \( \bar{q} = (\bar{x}_1, \ldots, \bar{x}_d) \) and \( q = (x_1, \ldots, x_n) \)

\[
\frac{\partial \lambda_i}{\partial x_j}(q) = \sum_{l=1}^{d} \frac{\partial \hat{\lambda}_i}{\partial \bar{x}_l}(\mathcal{F}_K^{-1}(q)) \frac{\partial \mathcal{F}_K^{-1}(q)}{\partial x_j}(q)
\]

From \([17]\), denoting \( \mathcal{E}_K = \mathcal{H}_K^{-1} \delta_K^T \in \mathcal{M}_{d,m}(\mathbb{R}) \) gives \( \frac{\partial \mathcal{F}_K^{-1}(q)}{\partial x_j}(q) = (\mathcal{E}_K)_{j,l} \). The barycentric coordinates are polynomials of first degree, so their gradients are constants and we obtain

\[
\nabla \lambda_i = \mathcal{E}_K \nabla \hat{\lambda}_i
\]

(in fact \( \mathcal{E}_K \) is the Jacobian matrix of \( \mathcal{F}_K^{-1} \)). The matrix \( \mathcal{H}_K \) is regular and symmetric, so \( \mathcal{E}_K = \delta_K \mathcal{H}_K^{-2} \) and we obtain

\[
\nabla \lambda_i = \delta_K \mathcal{H}_K^{-2} \nabla \hat{\lambda}_i.
\]

(19)
From (7), we deduced that

\[
\begin{pmatrix}
\nabla \lambda_1 \\
\vdots \\
\nabla \lambda_d
\end{pmatrix} = \mathbf{0}_d
\]

and thus

\[
\begin{pmatrix}
\nabla \lambda_1(q) \\
\vdots \\
\nabla \lambda_d(q)
\end{pmatrix} = \mathcal{A}_K \mathcal{H}_K^{-1}
\begin{pmatrix}
\nabla \lambda_1 \\
\vdots \\
\nabla \lambda_d
\end{pmatrix}
= \mathcal{A}_K \mathcal{H}_K^{-1}.
\]

As \( \sum_{i=0}^{d} \lambda_i(q) = 1 \), we immediately have

\[
\nabla \lambda_0(q) = - \sum_{i=1}^{d} \nabla \lambda_i(q).
\]

From (13) and (14), we immediately have:

**Remark 4** The gradients of the barycentric coordinates are linear combinations of \( q^1 - q^0, \ldots, q^d - q^0 \).

### 3.2 Integration over a d-simplex

If \( K \) is a non-degenerated \( d \)-simplex in \( \mathbb{R}^d \), from (16) we have \( \mathcal{J}_{\mathcal{F}_K}(\mathbf{q}) = \mathcal{A}_K \). Then \( \mathcal{A}_K \) is a regular square matrix and we have the classical formula:

\[
\int_K f(q)d\mathbf{q} = |\det(\mathcal{A}_K)| \int_K f \circ \mathcal{F}_K(\mathbf{q})d\mathbf{q}
\]

The following theorem extend this result to \( d \)-simplex in \( \mathbb{R}^n \), with \( 1 \leq d \leq n \).

**Theorem 5** Let \( K \subset \mathbb{R}^n \) be a non-degenerated \( d \)-simplex and \( f : K \rightarrow \mathbb{R} \).

\[
\int_K f(q)d\mathbf{q} = |\det(\mathcal{A}_K \mathcal{A}_K^T)|^{1/2} \int_K f \circ \mathcal{F}_K(\mathbf{q})d\mathbf{q}
\]

where \( K \) is the unit \( d \)-simplex in \( \mathbb{R}^n \), \( \mathcal{A}_K \in \mathcal{M}_{d,n}(\mathbb{R}) \) is defined by

\[
\mathcal{A}_K = \begin{pmatrix}
q^1 - q^0 \\
q^2 - q^0 \\
\vdots \\
q^d - q^0
\end{pmatrix}
\]

and \( \mathcal{F}_K : K \rightarrow K \) is given by

\[
\mathcal{F}_K(\mathbf{q}) = \mathcal{A}_K \mathbf{q} + \mathbf{q}^0
\]

**Proof:** The set \( \{v^1, \ldots, v^d\} \) is linearly independent so we can extend it to a basis \( \{v^1, \ldots, v^n\} \). We denote by \( \mathcal{A} \in \mathcal{M}_{n,n}(\mathbb{R}) \) the matrix such that the \( i \)-th column is the vector \( v^i \) for all \( i \in [1, n] \). So we have

\[
\mathcal{A} = \begin{pmatrix}
\mathcal{A}_K & v^{d+1} & \cdots & v^n
\end{pmatrix}
\]

By the QR-factorization theorem applied to the matrix \( \mathcal{A} \in \mathcal{M}_{n}(\mathbb{R}) \), there is an orthogonal matrix \( \mathcal{Q} \in \mathcal{M}_{n}(\mathbb{R}) \) and a regular upper triangular matrix \( \mathcal{R} \in \mathcal{M}_{n}(\mathbb{R}) \) such that

\[
\mathcal{A} = \mathcal{Q}\mathcal{R}
\]
So we have
\[ Q^t \mathbb{A} = \mathbb{R} \]
and we define the matrix \( \hat{\mathbb{A}} \in \mathcal{M}_{n,d}(\mathbb{R}) \) to be the first \( d \) columns of \( \mathbb{R}^n \):
\[ \hat{\mathbb{A}} = Q^t \mathbb{A}. \]

We can also note that
\[ \hat{\mathbb{A}} = \left( \begin{array}{c|c|c|c}
\mathbb{q}^1 - \mathbb{q}^0 & \mathbb{q}^2 - \mathbb{q}^0 & \cdots & \mathbb{q}^d - \mathbb{q}^0 \\
\hline
\end{array} \right) = \left( \begin{array}{c|c|c|c}
\mathbb{q}^1 & \mathbb{q}^2 & \cdots & \mathbb{q}^d \\
\hline
\end{array} \right). \]

Let \( \mathcal{F} : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be the bijective function defined by
\[ \mathcal{F}(\mathbf{x}) = Q^t (\mathbf{x} - \mathbb{q}^0) = \hat{\mathbf{x}} \quad (26) \]
and
\[ \mathbb{q}^i = \mathcal{F}(\mathbb{q}^i) = Q^t (\mathbb{q}^i - \mathbb{q}^0), \quad \forall i \in [0,d]. \]

By construction \( \mathbb{q}^0 = 0 \) and, \( \forall i \in [1,d], \mathbb{q}^i \) is the \( i \)-th column of the upper triangular matrix \( \mathbb{R} \). So we have
\[ \forall i \in [0,d], \quad \mathbb{q}^i \in \text{Vect}(\mathbf{e}^1, \ldots, \mathbf{e}^d) \]
where \{\mathbf{e}^1, \ldots, \mathbf{e}^n\} is the standard basis of \( \mathbb{R}^n \). The set \{\mathbb{q}^0, \ldots, \mathbb{q}^d\} are the vertices of the \( d \)-simplex \( \hat{K} = \mathcal{F}(K) \) and we deduce
\[ \hat{K} \subset \text{Vect}(\mathbf{e}^1, \ldots, \mathbf{e}^d). \quad (27) \]

By change of variables, we obtain
\[ \int_{\hat{K}} f(\mathbb{q}) d\mathbb{q} = \int_{\mathcal{F}(\hat{K})} f \circ \mathcal{F}^{-1}(\mathbb{q}) |\text{det}(J_{\mathcal{F}^{-1}}(\mathbb{q}))| d\mathbb{q} \]
where \( J_{\mathcal{F}^{-1}} \) is the Jacobian matrix of \( \mathcal{F}^{-1} \). From (26), we have \( J_{\mathcal{F}^{-1}}(\mathbb{q}) = \mathcal{Q} \) and as \( \mathcal{Q} \) is an orthogonal matrix, \( \text{det}(J_{\mathcal{F}^{-1}}(\mathbb{q})) = 1 \). So we obtain
\[ \int_{\hat{K}} f(\mathbb{q}) d\mathbb{q} = \int_{\mathcal{F}(\hat{K})} f \circ \mathcal{F}^{-1}(\mathbb{q}) d\mathbb{q}, \quad (28) \]

Let \( \mathcal{P} \in \mathcal{M}_{d,n}(\mathbb{R}) \) defined by
\[ \mathcal{P} = \left( \begin{array}{c|c|c}
\mathbf{I}_d & \mathbf{0}_{d,n-d} \\
\hline
\end{array} \right) \]
and
\[ \forall i \in [0,d], \quad \mathbb{q}^i = \mathcal{P} \mathbb{q}^i \in \mathbb{R}^d. \]

From (27), we deduce
\[ \forall i \in [0,d], \quad \mathbb{q}^i = \mathcal{P} \mathbb{q}^i = \left( \begin{array}{c}
\mathbb{q}^i_1 \\
\mathbf{0}
\end{array} \right). \]

Let \( \tilde{g} = f \circ \mathcal{F}^{-1} \) and \( \hat{K} \) be the \( d \)-simplex in \( \mathbb{R}^d \) with vertices \( \mathbb{q}^i, i \in [0,d] \). We denote by \( \mathcal{P} : \hat{K} \subset \mathbb{R}^d \rightarrow \hat{K} \subset \mathbb{R}^n \) the application defined by \( \mathcal{P}(\mathbb{q}) = \mathcal{P} \mathbb{q}^i \). We denote by \( g : \hat{K} \rightarrow \mathbb{R} \) the application defined by
\[ \tilde{g}(\mathbb{q}) = \tilde{g} \circ \mathcal{P}(\mathbb{q}) = \tilde{g} \left( \begin{array}{c}
\mathbb{q}^i_1 \\
\mathbf{0}
\end{array} \right). \]
So we obtain

\[
\int_K \hat{g}(q)dq = \int_K \check{g}(\check{q})d\check{q}
\]  
(29)

Let \( \check{A} \in \mathbb{M}_d(\mathbb{R}) \) be the matrix defined by

\[
\check{A} = \left( \check{q}^1 - \check{q}^0 ; \check{q}^2 - \check{q}^0 ; \ldots ; \check{q}^d - \check{q}^0 \right).
\]  
(30)

We can remark that

\[
\check{A} = P\check{A} \quad \text{and} \quad \check{A} = P^t\check{A}.
\]

Let \( \check{F} : \check{K} \rightarrow \check{K} \) be the bijective function defined by

\[
\check{F}(\check{q}) = \check{A}\check{q} + \check{q}^0.
\]

We can now apply the classical change of variables

\[
\int_K \hat{g}(q)dq = \int_K \hat{g} \circ \check{F}(\check{q}) |\det(\check{F}(\check{q}))|d\check{q}
\]
(31)

To resume from (22) and (29), we have

\[
\int_K f(q)dq = |\det(\check{A})| \int_K \hat{g} \circ \check{F}(\check{q})d\check{q}
\]

We can note that

\[
\check{g} \circ \check{F} = f \circ \check{F}^{-1} \circ P \circ \check{F}
\]

Let \( \check{F}_K = \check{F}^{-1} \circ P \circ \check{F} \), we have as expected

\[
\check{F}_K(\check{q}) = \check{F}^{-1} \circ P \circ \check{F}(\check{q})
\]

\[
= \check{F}^{-1}(P^\dagger(\check{A}\check{q}))
\]

\[
= \check{F}^{-1}(\check{A}\check{q})
\]

\[
= Q\check{A}\check{q} + \check{q}^0
\]

\[
= \check{A}_K\check{q} + \check{q}^0.
\]

and we obtain

\[
\int_K f(q)dq = |\det(\check{A})| \int_K f \circ \check{F}_K(q)dq
\]  
(32)

To obtain formula (22), it remains to prove that \(|\det(\check{A})| = |\det(\check{A}_K\check{A}_K)|^{1/2}\).

We have

\[
\check{A}_K^\dagger\check{A}_K = \check{A}_K^\dagger QQ^\dagger\check{A}_K
\]

\[
= \check{A}_K^\dagger \check{A}
\]

as \( \check{A}_K = Q\check{A} \)

\[
= \check{A}_K^\dagger P^\dagger P\check{A}
\]

as \( \check{A} = P^\dagger\check{A} \)

\[
= \check{A}_K^\dagger \check{A}
\]

as \( PP^\dagger = I_d \)

As \( \check{A} \) is a square matrix, we have \( \det(\check{A}^\dagger\check{A}) = \det(\check{A})^2 \) and thus

\[
|\det(\check{A})| = |\det(\check{A}_K\check{A}_K)|^{1/2}.
\]

\(\square\)
3.3 Volume of a d-simplex

The volume/measure of the d-simplex $K \subset \mathbb{R}^n$ is given by

$$|K| = \int_K 1 dq$$  \hspace{1cm} (33)

Using formula (22) with $f = 1$ gives

$$|K| = |\det(A_K^* A_K)|^{1/2} \int_K 1 dq = |\det(A_K^* A_K)|^{1/2} |\hat{K}|.$$  

From (9), we finally obtain

$$|K| = \frac{|\det(A_K^* A_K)|^{1/2}}{d!}.  \hspace{1cm} (34)$$

In [5] this formula is proved with geometrical arguments. We can also remark that if $d = n$ then $A_K$ is a square matrix and we obtain the classical formula

$$|K| = \frac{|\det(A_K)|}{d!}.  \hspace{1cm} (35)$$

3.4 Magic formula

In this section an exact computation of the integral over a d-simplex $K \subset \mathbb{R}^n$ for products of power of its barycentric coordinates given by (1) is proved by using previous results obtained by Laplace transforms.

Using formula (22) with $f(q) = \prod_{i=0}^d \lambda_{v_i}^\nu(q)$ gives

$$\int_K \prod_{i=0}^d \lambda_{v_i}^\nu(q) dq = |\det(A_K^* A_K)|^{1/2} \int_K \prod_{i=0}^d (\lambda_i \circ \mathcal{F}_K(q))^\nu_i dq$$

From (18) and (34), we obtain

$$\int_K \prod_{i=0}^d \lambda_{v_i}^\nu(q) dq = d! |K| \int_K \prod_{i=0}^d \hat{\lambda}_{v_i}^\nu(q) dq$$

Using formula (10) gives (1)

References


