The brane action for coherent ∞-operads

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L'action de membranes pour les ∞-opérades cohérentes

The brane action for coherent ∞-operads

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En fait, c'est cela la chose remarquable, quand on pose la question : « À quoi sert socialement la science ? », pratiquement personne n'est capable de répondre. Les activités scientifiques que nous faisons ne servent à remplir directement aucun de nos besoins, aucun des besoins de nos proches, de gens que nous puissions connaître. Il y a aliénation parfaite entre nous-même et notre travail.
Ce n'est pas un phénomène qui soit propre à l'activité scientifique, je pense que c'est une situation propre à presque toutes les activités professionnelles à l'intérieur de la civilisation industrielle. C'est un des très grands vices de cette civilisation industrielle.

Alexandre Grothendieck
Allons-continuer la recherche scientifique ?, 1972

Yesterday I found the courage at last to study your mathematical manuscripts even without reference books, and I was pleased to find that I did not need them. I compliment you on your work. The thing is as clear as daylight, so that we can't wonder enough at the way the mathematicians insist on mystifying it. But this comes from the one-sided way these gentlemen think.

Friedrich Engels
Letter to Karl Marx, 1881

Résumé

Cette thèse porte sur l'action de membranes, un mécanisme qui munit l'espace des extensions de l'opération identité dans une ∞-opérade cohérente $\mathcal{O}^{\otimes}$ d'une structure canonique de $\mathcal{O}$-algèbre dans l'∞-catégorie des cocorrespondances d'espaces.

Dans un premier temps, on démontre que la construction donnée par Mann–Robalo de cette action s'étend aux ∞-opérades cohérentes générales, sans restriction sur l'espace des couleurs ni sur celui des opérations unaires. On établit ensuite l'équivalence entre les modèles de Lurie et de Mann–Robalo de l'espace des extensions d'une opération, en les reliant par un zigzag explicite d'équivalences d'homotopie.

Dans le cas monochromatique, on démontre que, contrairement à ce que la littérature existante suppose, l'espace des extensions au sens de Lurie n'est en général pas équivalent à la fibre homotopique du morphisme d'oubli associé mais en est un quotient homotopique par l'action de l'∞-groupe des opérations unaires. Comme conséquence de ces résultats, on montre que les ∞-opérades de petits disques à repères tordues sont cohérentes et admettent une action de membranes reliée aux opérations de topologie des cordes.

Mots-clés

Action de membranes, opérades, ∞-opérades cohérentes, flèches tordues, ∞-catégorie des cocorrespondances, théorie des catégories supérieures, ensembles simpliciaux marqués, opérade des petits disques, topologie des cordes et des membranes, théories topologiques des champs.

Abstract

We study the brane action, which endows the space of extensions of the identity of a coherent ∞-operad $\mathcal{O}^{\otimes}$ with a canonical $\mathcal{O}$-algebra structure in the ∞-category of cospans of spaces.

First, we prove that Mann–Robalo's construction of the brane action extends to general coherent ∞-operads, with possibly multiple colors and non-contractible spaces of unary operations. Second, we establish that Lurie's model of the space of extensions of an operation is equivalent to Mann–Robalo's model, via an explicit zigzag of homotopy equivalences.

In the monochromatic case, contrary to what is claimed in existing literature, we show that the space of extensions in the sense of Lurie is not in general equivalent to the homotopy fiber of the associated forgetful morphism, but rather to its homotopy quotient by the ∞-group of unary operations. As a consequence of these results, we prove that the ∞-operads of B-framed little disks are coherent and admit brane actions related to string topology operations.

Keywords

Brane action, operads, coherent ∞-operads, twisted arrows, cospans, higher category theory, marked simplicial sets, little disks operad, string and brane topology, topological field theories.

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Contents

Chapter 1

Introduction

1.1 A glimpse of algebraic topology and operads

One of the fundamental goals of algebraic topology is to classify topological spaces, up to homotopy equivalence. The main tool to this end is the use of algebraic invariants, that is, objects endowed with operations whose algebraic structure encode the topological properties of the corresponding spaces. The simplest of these invariants is the homology of a topological space, which forms an abelian group, graded by the natural numbers.

While extremely useful, considering only the homology of a topological space does not in general retain all of its structure. To remedy this issue, algebraic topologists have studied refined versions of homology, in order to encode the homotopical properties of spaces more faithfully. These new invariants usually take the form of certain algebraic structures defined on chain complexes associated with the space. However, such algebraic structures can be intricate; the adequate language to define and study them is that of *operads*.

Operads

In a nutshell, an operad is a device that encapsulates all the operations that one can perform in any algebra of a given sort.

Consider for instance the associative operad, denoted Ass. It contains the information of all the possible ways one can multiply k inputs $a_1, \dots, a_k$ in an associative algebra A: these multiplicative operations are given by all the permutations on the symbols $a_1, \dots, a_k$. In other words, we may say that the set $\mathrm{Ass}(k)$ of arity k operations in the associative operad is in bijection with the symmetric group on k elements. These sets $\mathrm{Ass}(k)$, for varying $k \in \mathbb{N}$, are related one another by composition maps

$$- \circ_i -\colon \mathrm{Ass}(k) \times \mathrm{Ass}(m) \longrightarrow \mathrm{Ass}(k+m-1)$$

given by inserting an operation of arity k as the i-th input of an operation of arity m, thereby giving rise to a new operation of arity $k+m-1$.

An operad is then defined as a collection $\mathcal{O} = \{\mathcal{O}(k)\}_{k\in\mathbb{N}}$ of sets $\mathcal{O}(k)$ equipped with an action of the symmetric group Σ_k, together with a distinguished identity element $\mathrm{id} \in \mathcal{O}(1)$ and composition maps $- \circ_i - \colon \mathcal{O}(k) \times \mathcal{O}(m) \to \mathcal{O}(k+m-1)$ that are associative, unital and equivariant in an appropriate sense. The set $\mathcal{O}(k)$ encodes all the possible operations with k inputs in an $\mathcal{O}$-algebra.

More generally, one can replace sets and maps by topological spaces and continuous maps to obtain the notion of a *topological* operad. A similar definition gives operads in vector spaces, chain complexes, etc. This greater level of generality allows to consider new types of algebraic structures, where usual algebraic equations do not hold in a strict sense, but rather up to some homotopies, which in turn themselves satisfy some equations up to some *higher* homotopies, etc.

For example, the based loop space $\Omega_x X := \mathrm{Map}_*(S^1, X)$ of a pointed topological space (X, x) has a very natural algebraic structure given by concatenation of loops. This operation is not associative on the nose: the associativity equation holds only up to some homotopy given by reparametrization of the loops. The higher homotopies then encode higher coherences, in the sense of associativity-type relations between the various ways of concatenating multiple loops. The resulting algebraic structure, which in particular induces a group structure on the set of connected components $\pi_1(X, x)$ of $\Omega_x X$ (aka the fundamental group of X at x), is that of an $\mathbb{E}_1$-algebra.

Little disks operads

The topological operad $\mathbb{E}_1$ encoding the previous algebraic structure of the based loop space $\Omega_x X$ governs, more generally, the structure of all coherently homotopy-associative algebras. This operad $\mathbb{E}_1$ is the first of a sequence of topological operads $\mathbb{E}_n$, for $n \in \mathbb{N}^*$, whose corresponding algebras are associative up to homotopy and increasingly commutative up to homotopy, as n tends to infinity. The operad $\mathbb{E}_n$, originally introduced by Boardman–Vogt [BV73] and May [May72], is called the *operad of little disks of dimension* n and is of major importance in algebraic topology. The space $\mathbb{E}_n(k)$ of operations of arity k inside this operad is given by the space of configurations of k open disks of dimension n embedded in a larger such disk. Composition in the little disks operad is obtained by insertion of configurations of disks, as depicted in figure 1.1.

In our work, we do not consider topological operads, but instead the closely related notion of ∞-*operads*. While topological operads can be viewed as examples of ∞-operads (via a nerve construction), the latter notion is more flexible and adequate for the purposes of modern homotopy theory.

The little disks ∞-operads $\mathbb{E}_n$ form the paradigmatic examples of *coherent* ∞-operads. Together with the brane actions they give rise to, these ∞-operads are the central objects of this thesis. Other important examples of coherent ∞-operads come from geometry, most notably the operad $\mathcal{M}_{0,\bullet+1}$ of algebraic curves of genus 0 with marked points, as well as variants of its Deligne–Mumford compactification $\overline{\mathcal{M}_{0,\bullet+1}}$, which governs the structure of genus 0 Gromov–Witten invariants from enumerative geometry (see also the end of section 1.3).

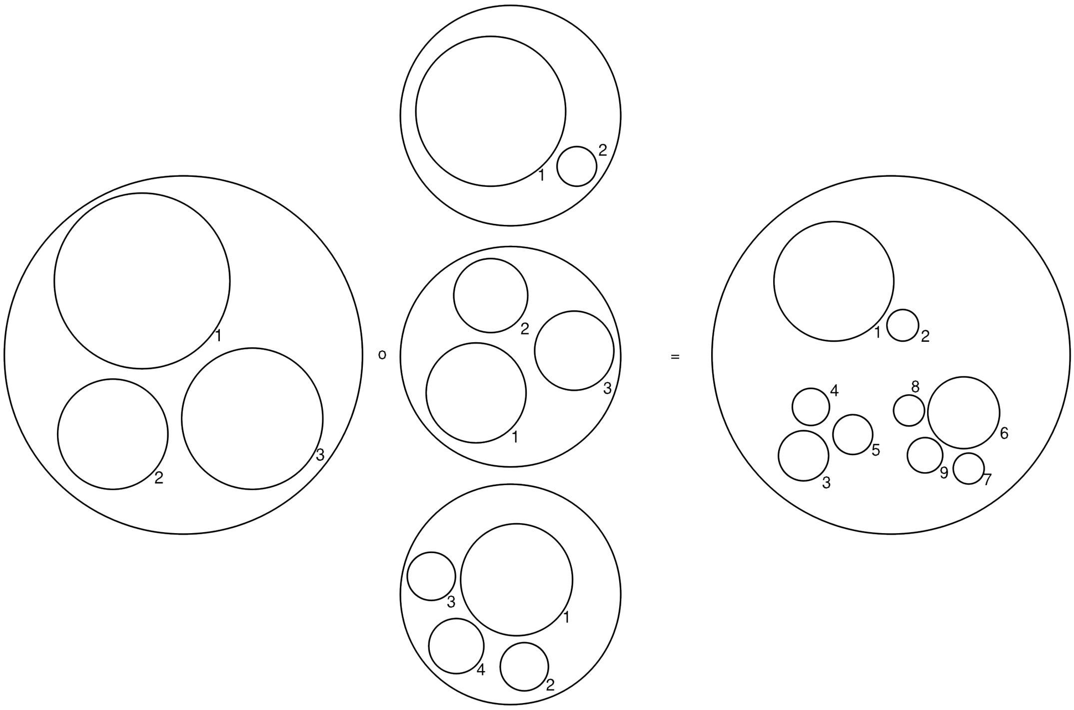


Figure 1.1: Composition in the little disks ∞-operad $\mathbb{E}_2$ of an operation of arity 3 (on the left) with three operations of arity 2, 3 and 4 (in the middle) yields an operation of arity 9 (on the right).

To motivate our study of coherent ∞-operads and their brane actions, we take a detour through string topology. We will then introduce the brane action in section 1.3 and explain our contribution in section 1.4.

1.2 String topology

For certain classes of spaces, the homology and its complex of singular chains naturally carry specific algebraic structures, on top of that of a graded abelian group. The previous example of the $\mathbb{E}_1$-algebra structure on the based loop space $\Omega_x X$ suggests to consider the related class of *free loop spaces*.

By the free loop space $\mathcal{L}X$ of a topological space X, we mean the space of continuous maps of the circle into X, endowed with the compact-open topology. It turns out that such spaces indeed have very rich algebraic structures, whose study has given rise to a subfield of algebraic topology named *string topology*.

Topological viewpoint

One of the roots of string topology can be tracked down to the investigations of the geometry of surfaces from the years 1980–1990's. A major contribution was Goldman's introduction and study of a Lie bracket on the free abelian group on isotopy classes of closed curves on a compact surface [Gol86], in relation to his celebrated work on the symplectic structure of character varieties [Gol84].

String topology started with the construction by Chas and Sullivan [CS99, CS04] of an associative product on the homology of the free loop space of a closed oriented manifold X, called the *loop product*, which restricts to the intersection product on the homology of X via the inclusion $X \to \mathcal{L}X$ of constant loops. Moreover, the interaction of this operation with the S^1-action induced by rotation of loops gives rise to a *Batalin–Vilkovisky-algebra* structure, which recovers Goldman's Lie algebra on $H_0(\mathcal{L}X)$ when X is a surface. The operad BV encoding this algebra is closely related to the little disks operad of dimension 2: a result of Getzler [Get94] identifies BV with the homology of the *framed* little disks operad $\mathbb{E}_2^{\mathrm{fr}}$, a variant of $\mathbb{E}_2$ obtained as a semi-direct product of the latter with the group $SO(2)$ of rotations.

One is led to wonder whether this BV-algebra is part of a larger structure and if moreover it can be lifted from homology to the level of the underlying chains. Motivated by such questions, the study of string topology has considerably expanded since Chas–Sullivan's seminal work, using methods from stable homotopy [CJ2f, BM19, Mor20, Roy13], combinatorial models of moduli spaces of Riemann surfaces [TZ06, God07, Kau07, Kup11, DPR15] or algebraic models based on Hochschild homology [Goo85, Jon87, Mer04, Mal11, GTZ12, Iri17, CHV22]. Certain string topology operations have also been extended to spaces beyond the case of manifolds, notably classifying groups [CM12, HL15], Gorenstein spaces [FT09] or oriented topological stacks [BGNX12].

Field theory viewpoint

This wealth of operations can be extended and organized into the structure of a topological field theory of dimension 2 (in a sense closely related to the original definition by Atiyah [Ati88] and Segal [Seg91]): from this perspective, operations on free loop spaces are induced by surfaces, viewed as cobordisms between their boundaries (as depicted in figure 1.2). Since the operad of framed little disks $\mathbb{E}_2^{\mathrm{fr}}$ can be realized as the moduli space of Riemann surfaces of genus 0 with boundaries, we can view the string topology BV-algebra as the genus 0 part of the homology of this topological field theory.

This viewpoint from field theory has been implemented in various forms [Cha05, CG04, CV06, Cos07, KS09, CTZ08, BCT09, WW16]. Let us mention Costello's approach, which is a form of the noncompact cobordism hypothesis in dimension 2 (see [Lur09b]): it consists in associating to every Calabi–Yau $\mathbb{E}_1$-algebra A a topological conformal field theory, that is, an action of chains of the moduli space of Riemann surfaces with boundary on the Hochschild homology of A. The string topology operations are then obtained by applying this result to the cochain complex $A = C^*(X)$ of the target manifold X, which is an $\mathbb{E}_1$-algebra with Calabi–Yau structure coming from the Poincaré duality pairing, and whose Hochschild homology is isomorphic to the cohomology of $\mathcal{L}X$, when X is simply connected (see [FTVP04]).

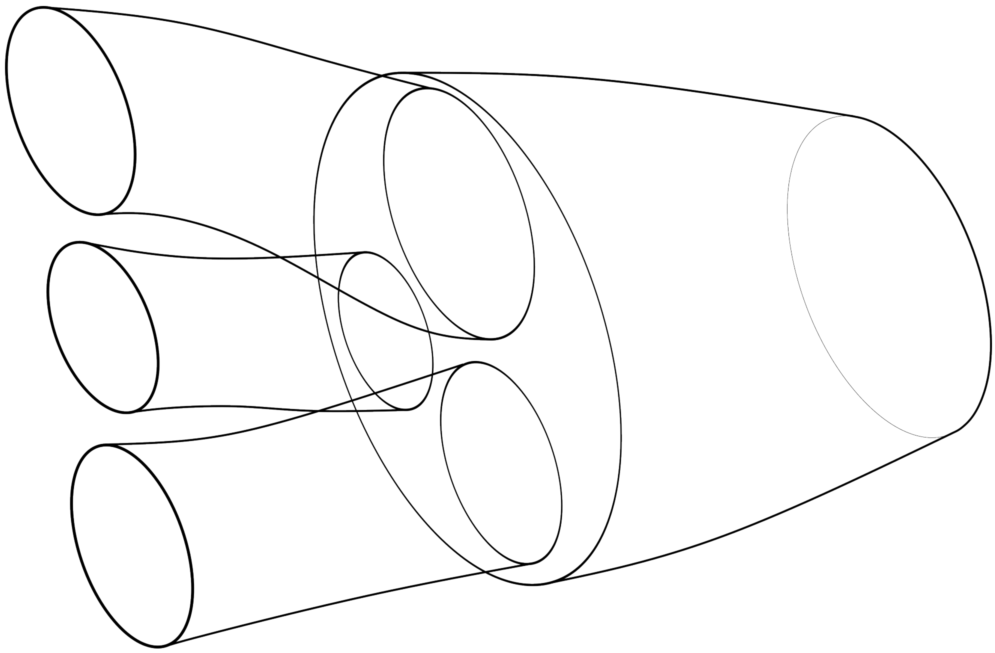


Figure 1.2: A configuration in $\mathbb{E}_2(3)$ giving rise to a cobordism between circles.

Moreover, the previous approach using topological conformal field theory makes apparent the strong analogy between string topology operations and the structure of Gromov–Witten invariants in enumerative geometry. This relation will come back in the next section, when discussing Toën and Mann–Robalo's approach [Toë13, MR18] to Gromov–Witten theory via the study of brane actions.

Symplectic viewpoint

Another aim of algebraic topology is to characterize geometric structures in terms of algebraic and homotopical data. This question constitutes a further motivation for the study of free loop spaces, since it has been conjectured that a full set of string topology operations on a closed oriented smooth manifold could encode part of its diffeomorphism type, beyond its underlying homotopy type [Sul07].

Such expectations come from the deep connections string topology possess with symplectic geometry. A central result in this vein is Viterbo's isomorphism [Vit98], as well as its generalization by Abouzaid [Abo15], which provides an isomorphism of BV-algebras between the homology of the free loops space of a closed oriented manifold X (twisted by a local system) and the so-called symplectic cohomology of its cotangent bundle T^*X. This relation is expected to be even stronger: for instance, Cieliebak and Latschev proposed in [CL09] (see also [CFL20]) that the sympletic field theory of the unit cotangent bundle of X and its equivariant string topology should form quasi-isomorphic structures of homotopy involutive Lie bialgebras, which are closely related to algebras over the operad $\mathbb{E}_2^{\mathrm{fr}}$ (see for instance [CMW16]).

Brane topology

String topology may be generalized to mapping spaces $\mathrm{Map}(S^n, X)$ from higher dimensional spheres: this field is often called *brane topology*.

For X a closed oriented manifold, Sullivan and Voronov have stated and sketched a proof that the shifted homology of $\mathrm{Map}(S^n, X)$ is an algebra over the higher dimensional version BV_{n+1} of the BV-operad, defined as the homology of the framed little disks operad $\mathbb{E}^{\mathrm{fr}}_{n+1}$ (as explained in Cohen–Voronov's book [CV06]). In particular, this homology inherits an $(n+1)$-Poisson algebra structure, that is, an algebra over the homology of the little disks operad $\mathbb{E}_{n+1}$. The commutative multiplication of these algebras has been constructed by Sullivan–Voronov and also appears in [Cha05, KS06, HKV06, BGNX12].

This motivates the following conjectural chain level generalization of Sullivan–Voronov's construction (which is implicit in [CV06, Section 5.4] and appears explicitly in the introduction of [GTZ12]).

Conjecture 1.2.1. *For X a closed oriented manifold, the chains on* $\mathrm{Map}(S^n, X)$ *form an algebra over the chains of the framed little disks operad* $\mathbb{E}^{\mathrm{fr}}_{n+1}$.

The case of the underlying $\mathbb{E}_{n+1}$-algebra structure has been proven by Ginot–Tradler–Zeinalian in [GTZ12], under the assumption that X is an n-connected Poincaré duality space whose homology groups are projective k-modules, where k is an arbitrary ring of coefficients for chains (see also [Hu06] for related results). Passing from $\mathbb{E}_{n+1}$ to $\mathbb{E}^{\mathrm{fr}}_{n+1}$ requires to incorporate the $SO(n+1)$-action on little disks, which is still an open problem.

One possible approach to the above conjecture is to realize brane topology operations via the mechanism of brane actions, which we now introduce.

1.3 Brane actions

Let us come back to Chas–Sullivan's loop product μ. Following a construction of Cohen and Jones [CJ2f] (completed in [Mor20]), one may construct μ from the following span diagram of spaces

$$\mathcal{L}X \times \mathcal{L}X \xleftarrow{\mathrm{in}} \mathrm{Map}(S^1 \vee S^1, X) \xrightarrow{\mathrm{out}} \mathcal{L}X \tag{1.1}$$

by a pull-push operation on homology, that is:

$$\mu = \mathrm{out}_* \circ \mathrm{in}^!.$$

Here, the map "out" is given by evaluation at the base point of the circle and the map "in" is the natural inclusion, which is of finite codimension, so that an *umkehr* (ou wrong-way) map $\mathrm{in}^!$ can be defined on homology [CK09].

As in the field theory viewpoint, diagrams of the form (1.1) are induced by certain cobordisms of surfaces, parametrized by the configurations of pairs of disks in $\mathbb{E}_2(2)$, as represented in figure 1.2.

Remarkably, such span diagrams arise canonically from the ∞-operad $\mathbb{E}_2$ itself. This results from a universal construction introduced by Toën in [Toë13], called the *brane action*, which can be loosely described as a formal incarnation of topological field theory structures in a general operadic context.

Toën's approach to brane actions

To describe this construction, let us first recall the notion of categories of cospans (see section 2.1). Given $\mathcal{C}$ an ∞-category with finite colimits, we may form its ∞-category of cospans, denoted $\mathrm{Cospan}(\mathcal{C})$, whose objects are those of $\mathcal{C}$ and whose morphisms from X_0 to X_1 are given by diagrams $X_0 \to Y \leftarrow X_1$, called *cospans*, where Y is some object of $\mathcal{C}$. Composition of cospans is given by taking pushouts, in the sense that a composite of $X_1 \to Y_{12} \leftarrow X_2$ with $X_0 \to Y_{01} \leftarrow X_1$ is given by

$$X_0 \to Y_{01} \amalg_{X_1} Y_{12} \leftarrow X_2.$$

Now let $\mathcal{O}^{\otimes}$ be an ∞-operad, which we assume to be monochromatic[1] for simplicity and suppose that $\mathcal{O}^{\otimes}$ is unital, that is, the space of nullary operations $\mathcal{O}(0)$ is contractible.

Given an operation σ of arity n, we define an *extension* of σ to be an operation σ^+ of arity $n+1$ that restricts to σ when forgetting the last input (up to some specified homotopy). More precisely, we consider the morphism $\mathcal{O}(n+1) \to \mathcal{O}(n)$ that forgets the last input, by composing with the identity on the first n inputs and with the unique nullary operation on the last one, and form the following ∞-fiber product (or homotopy pullback) of spaces

$$\begin{tikzcd} \mathcal{E}\mathrm{xt}_\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}(n+1) \arrow[d, "\text{forget}"] \\ * \arrow[r, "\sigma"] & \mathcal{O}(n). \end{tikzcd} \tag{1.2}$$

Definition 1.3.1. We refer to this space

$$\mathcal{E}\mathrm{xt}_\sigma = \mathcal{O}(n+1) \times^{\mathrm{h}}_{\mathcal{O}(n)} \{\sigma\}$$

as *Toën's model of the space of extensions of σ.*

Given two operations $\nu \in \mathcal{O}(n)$ and $\tau \in \mathcal{O}(m)$ and an index $i \in \{1, \dots, n\}$, composition at input i induces a cospan of spaces of extensions

$$\mathcal{E}\mathrm{xt}_\tau \xrightarrow{\nu \circ_i -} \mathcal{E}\mathrm{xt}_{\nu \circ_i \tau} \xleftarrow{- \circ_i \tau} \mathcal{E}\mathrm{xt}_\nu \tag{1.3}$$

well-defined in the homotopy category of spaces.

Now assume that the space $\mathcal{O}(1)$ of unary operations in the ∞-operad is contractible. In this situation, the space $\mathcal{E}\mathrm{xt}_{\mathrm{id}}$ of extensions of the identity operation

[1] Our ∞-operads are implicitly coloured, as in [Lur17]. By a monochromatic ∞-operad $\mathcal{O}^{\otimes}$, we then mean that $\mathcal{O}^{\otimes}$ has an essentially unique color.

is canonically equivalent to that of binary operations $\mathcal{O}(2)$. Let σ be an operation of arity n. Writing diagram (1.3) successively for $(\nu, \tau) = (\mathrm{id}, \sigma)$ and for $(\nu, \tau) = (\sigma, \mathrm{id}^{\oplus n})$ yields two composable cospans, with composite

$$\mathcal{O}(2)^{\amalg n} \xrightarrow[\text{ext. of inputs}]{\sigma \circ -} \mathcal{E}\mathrm{xt}_\sigma \xleftarrow[\text{ext. of outputs}]{- \circ \sigma} \mathcal{O}(2). \tag{1.4}$$

Informally, we may interpret the above diagram as expressing the following property: spaces of extensions come canonically equipped with particular elements of two types, coming either from extensions of the inputs or from extensions of the output. The mechanism of brane action then consists in assembling cospans (1.4) obtained for varying σ into the structure of an $\mathcal{O}$-algebra in the ∞-category $\mathrm{Cospan}(\mathcal{S})$ of cospans of spaces.

However, to ensure compatibility of the operadic structure with the composition of cospans, one needs to restrict to a certain class of ∞-operads, originally called *of configuration type* in [Toë13] and corresponding to the notion of *coherent ∞-operads* in more recent literature [Lur17]. We shall emphasize that the proof that these two notions indeed coincide was unavailable in the literature until our corollary 1.4.1, which requires the assumption that the space $\mathcal{O}(1)$ is contractible. We will come back to this question when adressing the closely related problem of comparing Toën's model $\mathcal{E}\mathrm{xt}_\sigma$ for spaces of extensions with Lurie's model $\mathrm{Ext}(\sigma)$ (defined in 1.3.4), at the end of this section (see problem C and also the discussion of section 5.1.2).

By [Toë13, Proposition 3.5], we may define ∞-operads of configuration type as follows.

Definition 1.3.2 (∞-operads of configuration type)**.** Let $\mathcal{O}^\otimes$ be a unital monochromatic ∞-operad with trivial space of unary operations. We say that $\mathcal{O}^\otimes$ is *of configuration type* if for every integers $n, m \geqslant 2$, every operations $\sigma \in \mathcal{O}(n)$, $\tau \in \mathcal{O}(m)$ and every integer $1 \leqslant i \leqslant n$, the canonical map

$$\mathcal{E}\mathrm{xt}_\tau \coprod_{\mathcal{O}(2)} \mathcal{E}\mathrm{xt}_\sigma \longrightarrow \mathcal{E}\mathrm{xt}_{\sigma \circ_i \tau}$$

is an equivalence.

The prototypical example of an ∞-operad of configuration type is given by the little disks ∞-operad $\mathbb{E}_{n+1}^\otimes$, for every $n \in \mathbb{N}$. This follows from the identification $\mathbb{E}_{n+1}(2) \simeq S^n$ and the equivalence, for every $\sigma \in \mathbb{E}_{n+1}(m)$,

$$\mathcal{E}\mathrm{xt}_\sigma = \mathbb{E}_{n+1}(m+1) \times^{\mathrm{h}}_{\mathbb{E}_{n+1}(m)} \{\sigma\} \simeq \bigvee^m S^n.$$

The construction of the brane action associated to an ∞-operad of configuration type is then given by the following result of Toën.

Theorem 1.3.3 ([Toë13])**.** *Let $\mathcal{O}$ be a unital monochromatic ∞-operad of configuration type, with contractible space of unary operations. Then the space $\mathcal{O}(2)$ of binary operations has a canonical $\mathcal{O}$-algebra structure in the ∞-category of cospans of spaces, with structure maps given by the cospan diagrams* (1.4)*.*

Operations on spaces of branes

As a consequence of the previous theorem, one can construct operations on mapping spaces $\mathrm{Map}(\mathcal{O}(2), X)$, by a pull-push procedure analogous to the one sketched at the beginning of this section for the loop product. An important feature of this construction is its level of generality: *indeed, it does not require X itself to be a topological space and can therefore be applied to various geometric contexts.*

Let $\mathcal{X}$ be an ∞-topos, which we think of as an ∞-category of geometric objects. Recall that there is a canonical functor $\mathcal{S} \to \mathcal{X}$ sending a space Z to the colimit of the constant diagram $Z \to \mathcal{X}$ with value the terminal object in $\mathcal{X}$. Through this functor, we can view $\mathcal{O}(2)$ as an object in $\mathcal{X}$ and transport its $\mathcal{O}$-algebra structure (given by the brane action) to the ∞-category $\mathrm{Cospan}(\mathcal{X})$.

Let X be an object in $\mathcal{X}$. The internal hom object $\mathrm{Map}(\mathcal{O}(2), X)$ in $\mathcal{X}$, called the *space of $\mathcal{O}$-branes on X* by Toën, carries an $\mathcal{O}$-algebra structure in $\mathrm{Span}(\mathcal{X})$, whose structural morphisms

$$\mathrm{Map}(\mathcal{O}(2), X)^n \xleftarrow{\text{out}} \mathrm{Map}\left(\mathcal{E}\mathrm{xt}_\sigma, X\right) \xrightarrow{\text{in}} \mathrm{Map}(\mathcal{O}(2), X), \tag{1.5}$$

for $\sigma \in \mathcal{O}(n)$, are obtained from the brane action by applying the functor $\mathrm{Map}(-, X)$.

In most applications, one is interested in inverting the "wrong-way" map in the above spans to obtain an $\mathcal{O}$-algebra structure in some more tractable, linear ∞-category. The general idea is as follows. Given a presentable stable monoidal $(\infty, 2)$-category $\mathcal{C}$, a functor $D\colon \mathcal{X} \to \mathcal{C}$ (which we think of as a linear invariant of objects in $\mathcal{X}$) that satisfies a certain base change condition and an object $X \in \mathcal{X}$ with some appropriate finiteness conditions, one can perform a pull-push operation (see [Ste20]) and obtain morphisms

$$D(\mathrm{Map}(\mathcal{O}(2), X))^{\otimes n} \xrightarrow{\text{out}_* \circ \text{in}^*} D(\mathrm{Map}(\mathcal{O}(2), X)) \tag{1.6}$$

that turn $D(\mathrm{Map}(\mathcal{O}(2), X))$ into an $\mathcal{O}$-algebra in $\mathcal{C}$.

Following Toën, we now describe an important example of this strategy in an algebro-geometric context. Let $\mathcal{X}$ be the ∞-topos $\mathrm{d}\mathcal{S}\mathrm{t}_k$ of derived stacks over a field k of characteristic 0. Consider the functor $D = \mathrm{QCoh}$ that assigns to every derived stack its derived ∞-category of quasi-coherent sheaves, viewed as an object of the $(\infty, 2)$-category $\mathcal{C} = \mathrm{dg}\mathcal{C}\mathrm{at}_k^{\mathrm{L}}$ of (possibly large) k-linear presentable dg-categories with functors preserving small colimits. For X a quasi-projective derived scheme, or more generally a perfect stack in the sense of [BZFN10] (see definition 5.4.19), the base change condition is satisfied and the brane action therefore yields an $\mathcal{O}$-algebra structure on the dg-category $\mathrm{QCoh}(\mathrm{Map}(\mathcal{O}(2), X))$ of quasi-coherent sheaves on the space of $\mathcal{O}$-branes on X.

In particular, for $\mathcal{O}^\otimes = \mathbb{E}_{n+1}^\otimes$ the ∞-operad of little disks of dimension $n+1$, one obtains an $\mathbb{E}_{n+1}$-algebra structure on the derived dg-categories of quasi-coherent sheaves on the space of branes $\mathrm{Map}(S^n, X)$ of X, for an important class of stacks X. Toën deduces from it a higher formality theorem, identifying the dg

Lie algebra associated to the $\mathbb{E}_{n+2}$-algebra of endomorphism of the unit object of $\mathrm{QCoh}(\mathrm{Map}(S^n, X))$ with that of shifted polyvector fields on X.

Program: string topology via brane actions

The work of Toën on operations on spaces of branes naturally suggest the following approach to string and brane topology, as well as further generalizations beyond the realm of manifolds.

- Can one adapt the linearization strategy described above to the topological setting in order to prove conjecture 1.2.1, thereby extending Sullivan–Voronov's construction to the chain level?
- Can one develop brane topology operations, including the original string topology ones, in more general geometric contexts, such as those of derived differentiable and derived algebraic stacks, and relate them?

Note that the framed little disks ∞-operad $\mathbb{E}_{n+1}^{\mathrm{fr}}$ appearing in conjecture 1.2.1 has a non-contractible space of unary operations $\mathbb{E}_{n+1}^{\mathrm{fr}}(1) \simeq SO(n+1)$, so that Toën's theorem 1.3.3 does not apply to this case. The first step towards realizing program 1.3 is therefore to extend the brane action to encompass the cases of ∞-operads with non-contractible spaces of unary operations. Moreover, to incorporate module-type structures into the brane action, one would like to drop the requirement for the input ∞-operad $\mathcal{O}^{\otimes}$ to be monochromatic in the construction of brane actions.

Problem A. *Extend the brane action to general ∞-operads of configuration type, with possibly multiple colors and non-contractible space of unary operations.*

The first contribution of this thesis is to provide a solution to this problem (see theorem A).

The above program is further motivated by the analogous situation of Gromov–Witten invariants, for which brane actions turned out to be particularly relevant [MR18].

Gromov–Witten invariants

Since Gromov–Witten theory will play no role in this work but a motivational one, we only give a very sketchy introduction to these ideas.

Given a smooth projective algebraic variety X over $\mathbb{C}$ and some subvarieties of X, one can associate rational numbers, called *Gromov–Witten invariants*, which have an enumerative interpretation in terms of maps from stable curves of prescribed genus to X, transverse to the chosen subvarieties. These invariants, introduced by Kontsevich and Manin [KM94] in the context of algebraic geometry, can be encoded using different structures: quantum products, cohomological field theories and Frobenius manifolds, among others. This led Manin and Toën to the idea that the Gromov–Witten invariants of X could be detected at the level of the derived category of X.

The construction of categorified Gromov–Witten invariants was then one of the major motivation for Toën's work on brane actions. This was accomplished by Mann and Robalo in [MR18], for the genus 0 situation, by applying the brane action to variants of the ∞-operad $\{\overline{\mathcal{M}_{0,n+1}}\}_{n\in\mathbb{N}}$ of stable algebraic curves of genus 0 with marked points. The strength of this method is that *invariants are constructed at a purely geometric - or motivic - level*, in the sense that the structure exists before taking any invariant, such as cohomology or K-theory.

Mann–Robalo's approach

The approach taken in [MR18] (see also the survey [MR21]) relies on a new construction of the brane action, very different from Toën's original one, and will be presented in details in section 2.4. For the moment, let us simply note that their definition of the brane action is encapsulated as an explicit fibration

$$\pi\colon \mathcal{B}\mathcal{O} \longrightarrow \mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes}$$

over the twisted arrow ∞-category of the symmetric monoidal envelope of $\mathcal{O}^{\otimes}$, whose classifying functor gives the desired $\mathcal{O}$-algebra structure in cospans of spaces. We will call this functor π the *brane fibration*.

On the one hand, Toën's definition of the brane action uses the model of Segal operads for ∞-operads and relies on model categorical and strictification arguments, which have the drawback of making the resulting construction rather inexplicit. On the other hand, Mann–Robalo's work is phrased in the language of quasicategories and involves Lurie's specific model of ∞-operads [Lur17], but has nevertheless the advantage of coming close to a model-independent construction.

However, contrary to Toën's original approach, Mann and Robalo do not consider ∞-operads of configuration type, but instead the analogous notion of *coherent* ∞-operads in the sense of Lurie, implicitly identifying these two definitions without proof.

The definition of coherence for ∞-operads relies on modeling the spaces of extensions of an operation σ via an explicit simplicial set $\mathrm{Ext}(\sigma)$ (see definition 2.2.3[2]) that we shall call *Lurie's model of the space of extensions* of σ. Following Lurie, we can now informally define coherence as follows.

Definition 1.3.4 (Coherent ∞-operads)**.** Let $\mathcal{O}^{\otimes}$ be an ∞-operad, with possibly several colors and without any assumption on the space of unary operations. We say that $\mathcal{O}^{\otimes}$ is *coherent* if it is unital, its underlying ∞-category is a Kan complex and moreover for every composable operations $f\colon X \to Y$ and $g\colon Y \to Z$, the

[2] Our definition of the simplicial set $\mathrm{Ext}(\sigma)$ and therefore the corresponding definition of coherent ∞-operads differ slightly from that of [Lur17]. We refer to remark 2.2.5 for a justification of this difference.

diagram

$$\begin{tikzcd} \mathrm{Ext}(\mathrm{id}_Y) \arrow[r] \arrow[d] & \mathrm{Ext}(g) \arrow[d] \\ \mathrm{Ext}(f) \arrow[r] & \mathrm{Ext}(g \circ f), \end{tikzcd} \tag{1.7}$$

which is well-defined in the homotopy category of spaces, is homotopy cocartesian. We refer to definition 2.2.6 for a more rigorous expression of the above condition.

The construction of the brane action given in [MR18, Theorem 2.1.7] then takes the form of the following statement, analogous to Toën's theorem 1.3.3: for $\mathcal{O}^{\otimes}$ a coherent monochromatic ∞-operad with $\mathcal{O}(0) \simeq \mathcal{O}(1) \simeq *$, there exists a map of ∞-operads

$$\mathcal{O}^{\otimes} \longrightarrow \mathrm{Cospan}(\mathcal{S})^{\otimes}$$

that sends the color $c \in \mathcal{O}$ to the space $\mathrm{Ext}(\mathrm{id}_c)$ and an operation $\sigma\colon X \to Y$ to a cospan

$$\mathrm{Ext}(\mathrm{id}_X) \longrightarrow \mathrm{Ext}(\sigma) \longleftarrow \mathrm{Ext}(\mathrm{id}_Y). \tag{1.8}$$

Note that Mann–Robalo's construction actually relies on *yet another model* for the spaces of extensions of an operation σ, given by the fiber $\mathcal{B}\mathcal{O}_\sigma$ of the brane fibration they define. More precisely, the above theorem of Mann–Robalo requires an identification of $\mathcal{B}\mathcal{O}_\sigma$ with Lurie's model $\mathrm{Ext}(\sigma)$, but the proof of this fact is left unexplicit in [MR18].

However, it seems to the author that no straightforward comparison between those two definitions is available. For instance, simply writing an explicit morphism of simplicial sets relating the two models already seems a non-trivial problem. We are therefore left with the following issue.

Problem B. *Given an operation σ in a unital ∞-operad, prove the equivalence between Mann–Robalo's model $\mathcal{B}\mathcal{O}_\sigma$ and Lurie's model $\mathrm{Ext}(\sigma)$ parametrizing extensions of σ.*

To apply brane actions to particular examples of coherent ∞-operads, or to prove that a given ∞-operad is coherent, one needs to compute the spaces of extensions. For that purpose, Mann–Robalo's model $\mathcal{B}\mathcal{O}_\sigma$ and Lurie's $\mathrm{Ext}(\sigma)$ are both highly impractical. Identifiying the homotopy type of the space of extensions $\mathrm{Ext}(\mathrm{id})$ supporting the brane action seems unnecessarily difficult if using only the definition of $\mathcal{B}\mathcal{O}_\sigma$ and $\mathrm{Ext}(\sigma)$, even in simple examples such as that of the little disks ∞-operad $\mathbb{E}_n$.

In particular, we have the following pair of problems.

Problem C. *Provide a method to compute spaces of extensions in particular examples of ∞-operads.*

Problem D. *Prove that the framed little disks ∞-operad $\mathbb{E}_n^{\mathrm{fr}}$ is coherent.*

In this thesis, we will solve problems B, C and D, via the corresponding theorems B, C and D.

Kern's work

Finally, let us also mention the recent work of Kern [Ker21], who extended Mann–Robalo's proof of the brane action to the case of colored ∞-operads, satisfying a weaker form of unitality[3] and without the assumption that $\mathcal{O}(1)$ is contractible.

Let us note that, in addition to further applications to Gromov–Witten theory, Kern explains how in absence of the coherence assumption on $\mathcal{O}^{\otimes}$, the brane action takes the form of a lax morphism of categorical ∞-operads $\mathcal{O}^{\otimes} \to \mathrm{Cospan}(\mathcal{S})$, a result already present in Toën's original paper. The approach taken by Kern is phrased in terms of the *algebraic patterns* introduced by Chu–Haugseng in [CH21]. As a benefit of this high level of generality, and even if the author has to restrict eventually to the particular case of the algebraic pattern encoding ∞-operads, his work paves the way towards generalizations of the brane action for a larger class of algebraic patterns.

However, Kern's work takes for granted the equivalence between the spaces $\mathrm{Ext}(\sigma)$ and $\mathcal{B}\mathcal{O}_{\sigma}$, so that his proof is confronted with the same issue as Mann–Robalo's, namely problem B, which is then solved by our theorem B.

1.4 Main results

Extension of the brane action to general coherent ∞-operads

The first contribution of this thesis is to extend the mechanism of brane operations to encompass the case of general coherent ∞-operads, without any restrictions on the space of colors or that of unary operations, thereby generalizing Toën's theorem 1.3.3 and solving problem A.

Theorem A. *Let $\mathcal{O}^{\otimes}$ be a coherent ∞-operad. Then the collection of spaces $\mathrm{Ext}(\mathrm{id}_X)$, for varying colors $X \in \mathcal{O}$, carries a canonical $\mathcal{O}$-algebra structure in $\mathrm{Cospan}(\mathcal{S})$, with structural maps given by cospan diagrams* (1.8)*.*

Our approach is based on Mann–Robalo's construction and relies on a careful analysis of the brane fibration $\pi\colon \mathcal{B}\mathcal{O} \to \mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes}$. It was somewhat unexpected that the assumption of contractibility of the space of unary operations can simply be dropped from the theorem, since both Mann–Robalo's and Toën's proofs seem to make essential use of this hypothesis.

Comparison of models of spaces of extensions

Our second main result provides a solution to problem B. In other words, we prove the following statement.

[3] The precise condition, called *hapaxunitality* in [Ker21, Definition 2.2.1.2.8], requires that the ∞-operad has a distinguished color whose ∞-groupoid of unary endomorphisms is contractible.

Theorem B (Theorem 4.1.1)**.** *Let σ be an active morphism in a unital ∞-operad $\mathcal{O}^{\otimes}$. Then the fiber $\mathcal{BO}_\sigma$ of the brane fibration and the ∞-category of extensions* $\mathrm{Ext}(\sigma)$ *are equivalent.*

Note that this result is actually necessary in Mann–Robalo's approach of the brane action, and therefore also in our proof of theorem A (as well as in Kern's approach). Our strategy to prove theorem B consists in providing an explicit, ad-hoc zigzag of homotopy equivalences between $\mathcal{BO}_\sigma$ and $\mathrm{Ext}(\sigma)$.

We now turn to our solution to problem C.

Recall that in order to compute the homotopy type of the spaces of extensions in applications, neither of the models $\mathrm{Ext}(\sigma)$ or $\mathcal{BO}_\sigma$ of Lurie and Mann–Robalo is practical. On the other hand, Toën's model $\mathcal{O}(n+1) \times^{\mathrm{h}}_{\mathcal{O}(n)} \{\sigma\}$ is very suitable to computations in particular examples, and indeed all known computations involving spaces of extensions rely on the equivalence with Toën's definition.

Such an equivalence for Lurie's model $\mathrm{Ext}(\sigma)$ (and therefore also for Mann–Robalo's model, by theorem B) is claimed in [Lur17, Section 5.1.1]. More precisely, a comparison map is defined and asserted to be an equivalence. However, we find that Lurie's model $\mathrm{Ext}(\sigma)$ only agrees with Toën's when the ∞-operad $\mathcal{O}^{\otimes}$ has a contractible space of unary operations. Moreover, we provide a counter-example when this assumption fails, thereby contradicting the corresponding statement in [Lur17]. We refer to section 5.1.2 for a more detailed discussion.

The general situation is explained by the following result, which exhibits $\mathrm{Ext}(\sigma)$ as a quotient of $\mathcal{E}\mathrm{xt}_\sigma$ by an $\mathcal{O}(1)$-action.

Theorem C (Theorem 5.1.1)**.** *Let $\mathcal{O}^{\otimes}$ be a monochromatic unital ∞-operad whose underlying ∞-category $\mathcal{O}$ is an ∞-groupoid and let $\sigma \in \mathcal{O}(n)$ an operation of arity n. Choose a semi-inert morphism $i\colon \langle n\rangle \to \langle n+1\rangle$ in $\mathcal{O}^{\otimes}$. Then the space* $\mathrm{Ext}(\sigma)$ *is equivalent to the homotopy quotient of $\mathcal{O}(n+1) \times^{\mathrm{h}}_{\mathcal{O}(n)} \{\sigma\}$ by an action of the ∞-group $\mathcal{O}(1)$ of unary operations on the additional color of the extensions.*

As a direct consequence of this theorem, we justify that configuration type and coherent ∞-operads agree, at least in absence of non-trivial unary operations.

Corollary 1.4.1. *Let $\mathcal{O}^{\otimes}$ be a monochromatic ∞-operad with $\mathcal{O}(1) \simeq *$. Then $\mathcal{O}^{\otimes}$ is coherent if and only if it is of configuration type.*

Recall that the ∞-operad of little disks $\mathbb{E}^{\otimes}_{n+1}$ is coherent for any $n \geqslant 0$, by Lurie's result [Lur17, Theorem 5.1.1.1], whose proof relies on the validity of our theorem C.

Using the computation tool given by the previous theorem, we extend this coherence result to the variants $\mathbb{E}^{\otimes}_B$ of $\mathbb{E}^{\otimes}_{n+1}$ obtained by endowing disks with a framing datum (see [AF15]). These ∞-operads depend on the choice of a Kan complex B equipped with a Kan fibration $B \to \mathsf{BTop}(n+1)$ to the classifying space of the topological group of self-homeomorphisms of $\mathbb{R}^{n+1}$; one recovers the case of framed little disks $\mathbb{E}_{n+1}^{\mathrm{fr}}$ by taking $B = \mathrm{B}SO(n+1)$. Thus, the following general result solves problem D.

Theorem D (Theorem 5.4.8)**.** *Let B a Kan complex equipped with a Kan fibration to $\mathrm{B}\mathsf{Top}(n+1)$. Then the ∞-operad of B-framed little disks $\mathbb{E}_B^{\otimes}$ is coherent.*

One can prove that the space of extensions $\mathrm{Ext}(\mathrm{id}_b)$ of any color $b \in B$ is homotopy equivalent to the sphere S^n. As a consequence of theorems D and A, we obtain an $\mathbb{E}_B$-algebra structure on S^n in cospans of spaces.

Corollary 1.4.2. *Let X be a topological space. Then the space of branes $\mathrm{Map}(S^n, X)$ has an $\mathbb{E}_B$-algebra structure in $\mathrm{Span}(\mathcal{S})$ given by the brane action.*

Taking $B = \mathrm{B}SO(n+1)$, this yields an $\mathbb{E}_{n+1}^{\mathrm{fr}}$-algebra structure on the brane space $\mathrm{Map}(S^n, X)$ in the ∞-category of spans of spaces, hence proving the conjecture 1.2.1 *at the level of spans* and thereby providing a first step in the realization of the general program 1.3.

1.5 Outline of the thesis

We start in chapter 2 by recalling some important constructions: the ∞-categories of spans and that of twisted arrows, the precise definition of Lurie's model $\mathrm{Ext}(\sigma)$ for the space of extensions and the definition of coherent ∞-operads. We then define the brane fibration, following Mann–Robalo, and outline the proof of theorem A. This proof is then completed in chapter 3, by establishing that the functor $\pi\colon \mathcal{B}\mathcal{O} \to \mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes}$ is indeed a cartesian fibration (theorem 2.5.1).

Chapter 4 is devoted to the proof of B, that is the comparison between Mann–Robalo's and Lurie's model of spaces of extensions, via the construction of an explicit zigzag of homotopy equivalences.

Finally, we deal in chapter 5 with the problem of computing the homotopy type of spaces of extensions, by establishing an equivalence between Toën's and Lurie's models, thereby proving theorem C. Moreover, we discuss how our results differ from a claim in [Lur17] and provide a counterexample to the latter statement. The end of chapter 5 concerns applications to string topology, via a proof of coherence of the ∞-operad of B-framed little disks (theorem D). We end with a discussion of the new operations on spaces of branes that the previous result allows to construct, both in the topological context (at the span level) and for derived algebraic stacks (at the level of derived categories).

An appendix gathers some auxiliary definitions and results that are used throughout the thesis. Most notably, we prove some results concerning marked anodyne morphisms, which to the knowledge of the author do not appear in the literature and might be of independent interest.

1.6 Notations and conventions

- We work in the particular model of ∞-category theory given by quasicategories and use Lurie's presentation of ∞-operads. Our notations generally follow those of [Lur09a] and [Lur17].
- **Particular arrows:** monomorphisms are denoted as $A \hookrightarrow B$, cofibrations as $A \rightarrowtail B$ and atomic morphisms (see definition 2.2.1) as $A \lhook\joinrel\rightharpoonup B$.
- When considering a diagram $X\colon P \to \mathcal{C}$ from a poset P to an ∞-category $\mathcal{C}$ and a sequence $i_0 \leqslant i_1 \leqslant \ldots \leqslant i_n$ in P, we will write $X_{i_0} \ldots X_{i_n}$ for the n-simplex of $X \circ \langle i_0 \ldots i_n \rangle\colon \Delta^n \to P \to \mathcal{C}$. For instance, the notation $X_i X_j$ denotes the unique morphism $X_i \to X_j$ of the diagram.
- Given a finite linear order $I = \{i_0 < i_1 < \cdots < i_n\}$, the full subsimplex of Δ^I on the objects $i_{j_0} < \cdots < i_{j_k}$ will be denoted $\Delta^{i_{j_0} \cdots i_{j_k}}$ (unless $k = 0$). Similarly, $\Lambda^{i_{j_0} \cdots i_{j_k}}_{i_{j_p}}$ stands for the horn in $\Delta^{i_{j_0} \cdots i_{j_k}}$ obtained by removing the face opposed to vertex i_{j_p}. For instance, the horns Λ^{12}_1 and Λ^{12}_2 are respectively the simplicial subsets $\Delta^{\{2\}}$ and $\Delta^{\{1\}}$ of the 1-simplex Δ^{12}, while the notation Λ^{12}_0 does not make sense in our convention.
- For simplicity, given an ∞-operad $\mathcal{O}^{\otimes}$, we will often write $\mathcal{E}$ for its symmetric monoidal envelope $\mathrm{Env}(\mathcal{O})^{\otimes}$ and $\mathcal{T}$ for the associated twisted arrows ∞-category $\mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes}$ (see notation 2.3.1).
- We let $\mathbb{F}_*$ denote the nerve of the category of pointed finite sets. We usually identify $\mathbb{F}_*$ with its equivalent full subcategory on the pointed sets $\langle n \rangle = (\{0, \ldots, n\}, 0)$.

Chapter 2

The brane fibration

In this chapter, following [MR18], we explain how the brane action of theorem A arises from a certain fibration, which we call *the brane fibration*. Before giving the precise construction, we recall the notions of ∞-categories of (co)spans, of twisted arrows, of spaces of extensions in the sense Lurie and the definition of coherent ∞-operads.

Contents

2.1 Categories of spans and of twisted arrows

Given an ∞-category $\mathcal{C}$ with finite limits, we may form the ∞-category $\mathrm{Span}(\mathcal{C})$ of spans in $\mathcal{C}$, whose objects are those of $\mathcal{C}$, morphisms between two objects X and Y are given by span diagrams $X \longleftarrow Z \longrightarrow Y$ and composition is given by taking pullback (see [Bar13] or [Hau18] for a rigorous ∞-categorical definition). Dually, if $\mathcal{C}$ has finite colimits, we may consider its ∞-category of cospans $\mathrm{Cospan}(\mathcal{C})$ defined as $\mathrm{Span}(\mathcal{C}^{\mathrm{op}})$.

The ∞-category $\mathrm{Span}(\mathcal{C})$ has a canonical symmetric monoidal structure $\mathrm{Span}(\mathcal{C})^{\otimes_\times}$ induced from the cartesian monoidal structure on $\mathcal{C}^\times$, although $\mathrm{Span}(\mathcal{C})^{\otimes_\times}$ is not itself cartesian.

Definition 2.1.1 (Twisted arrow ∞-category)**.** Let $s\colon \Delta \to \Delta$ be the functor given by $s[n] = [n] * [n]^{\mathrm{op}}$. Precomposition with s yields an endofunctor $s^*\colon \mathsf{sSet} \to \mathsf{sSet}$ that we shall denote Tw. Left Kan extension of s along the Yoneda embedding of Δ induces a functor $s_*\colon \mathsf{sSet} \to \mathsf{sSet}$ left adjoint to

Tw. The image under Tw of an ∞-category $\mathcal{C}$ is again an ∞-category $\mathrm{Tw}(\mathcal{C})$ called its *twisted arrow* ∞-category, whose n-simplices are $(2n+1)$-simplices $s_*(\Delta^n) = \Delta^n * \Delta^{n,\mathrm{op}} \to \mathcal{C}$, represented as

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d] & X_1 \arrow[r] \arrow[d] & \dots \arrow[r] & X_n \arrow[d] \\ \overline{X_0} & \overline{X_1} \arrow[l] & \dots \arrow[l] & \overline{X_n}. \arrow[l] \end{tikzcd}$$

To depict a morphism in $\mathrm{Tw}(\mathcal{C})$, that is, a twisted arrow between two arrows f and g of $\mathcal{C}$, we will often write $f \rightsquigarrow g$.

Remark 2.1.2. Given a 2-simplex σ

$$\begin{tikzcd} & Y \arrow[dr, "g"] & \\ X \arrow[ur, "f"] \arrow[rr, "h"] & & Z \end{tikzcd}$$

in $\mathcal{C}$ that exhibits h as a composite of g and f, we obtain twisted arrows $h \rightsquigarrow g$ and $h \rightsquigarrow f$ in $\mathrm{Tw}(\mathcal{C})$ given respectively by the following commutative squares:

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] \arrow[dr, "h"] & Z \\ Y \arrow[r, "g"'] & Z \arrow[u, "\mathrm{id}_Z"'] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\mathrm{id}_X"'] \arrow[dr, "f"] & Z \\ X \arrow[r, "f"'] & Y \arrow[u, "g"'] \end{tikzcd}$$

in which the 2-simplices are either degenerate, are equal to σ.

By [Lur17, Example 5.2.2.23.], any symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ induces a symmetric monoidal structure $\mathrm{Tw}(\mathcal{C})^{\otimes}$ on the twisted arrow ∞-category $\mathrm{Tw}(\mathcal{C})$, in which the tensor product of two morphisms $f\colon x \to y$ and $g\colon z \to t$ is the obvious arrow of the form $f \otimes g\colon x \otimes z \to y \otimes t$.

An important feature of the construction of ∞-category of twisted arrows is the following universal property.

Proposition 2.1.3 (Universal property of Tw and Span)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be two ∞-categories and assume that $\mathcal{D}$ has all finite limits. Then:*

(1) There is a natural equivalence between the space of functors $\mathcal{C} \to \mathrm{Span}(\mathcal{D})$ and that of functors $F\colon \mathrm{Tw}(\mathcal{C}) \to \mathcal{D}$ satisfying the pullback condition*: namely that for every 2-simplex $h\colon X \xrightarrow{f} Y \xrightarrow{g} Z$ exhibiting h as a composite of g and f, the induced square*

$$\begin{tikzcd} h \arrow[r, rightsquigarrow] \arrow[d, rightsquigarrow] & g \arrow[d, rightsquigarrow] \\ f \arrow[r, rightsquigarrow] & \mathrm{id}_Y \end{tikzcd}$$

in $\mathrm{Tw}(\mathcal{C})$ is sent by F to a cartesian square in $\mathcal{D}$.

(2) If $\mathcal{C}^{\otimes}$ is a symmetric monoidal ∞-category with underlying ∞-category $\mathcal{C}$, then there is a natural equivalence between the space of symmetric monoidal functors $\mathcal{C}^{\otimes} \to \mathrm{Span}(\mathcal{D})^{\otimes\times}$ and that of symmetric monoidal functors $\mathrm{Tw}(\mathcal{C})^{\otimes} \to \mathcal{D}^{\times}$ satisfying the above pullback condition.

A proof of the first part of this result can be found in the appendix of [Ras14, Section 20]: there, the statement takes the stronger form of an adjunction between Cat_{∞} and a certain ∞-category $\mathsf{Cat}_{\infty}^{\mathrm{dir}}$ of small ∞-categories with directions, which fully-faithfully contains the ∞-category of small ∞-categories with finite limits and functors preserving them. In particular, this requires to enhance Tw to a functor $\mathsf{Cat}_{\infty} \to \mathsf{Cat}_{\infty}^{\mathrm{dir}}$. The extension to the symmetric monoidal case is explained in [MR18, Corollary 2.1.3.].

2.2 Extensions and coherent ∞-operads

In this subsection, we recall the definition of the ∞-category of extensions of an operation in an ∞-operad and the closely related notion of coherence. We essentially follow [Lur17, Section 3.3.1], except for a small difference in the definition of $\mathrm{Ext}(\sigma)$ (see remark 2.2.5).

Let $p\colon \mathcal{O}^{\otimes} \to \mathbb{F}_*$ be a unital ∞-operad.

Definition 2.2.1 (Semi-inert and atomic maps)**.** Let $f\colon X \to Y$ be a morphism in $\mathcal{O}^{\otimes}$, corresponding to a morphism $\alpha = p(f)\colon \langle n\rangle \to \langle m\rangle$ in $\mathbb{F}_*$ together with a family of multimorphisms $f_j\colon \{X_i\}_{\alpha(i)=j} \to Y_j$ for $j \in \langle m\rangle^{\circ}$. We say that f is *semi-inert* if for every $j \in \langle m\rangle^{\circ}$

- either the set $\alpha^{-1}\{j\}$ is empty, or
- the set $\alpha^{-1}\{j\}$ is the singleton $\{i_j\}$ and the map $f_j\colon X_{i_j} \to Y_j$ is an equivalence.

Following the terminology of [Ker21], we say that f is *atomic* if it is semi-inert and lies over an inclusion $\alpha\colon \langle n\rangle \to \langle n+1\rangle$. In other words, f is atomic if and only if it is semi-inert with no non-trivial factorization through another semi-inert morphism. Given a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, hook, "f_X"'] & Y \arrow[d, hook, "f_Y"] \\ X' \arrow[r, "f"] & Y' \end{tikzcd} \qquad \text{or} \qquad \begin{tikzcd} & X \arrow[dr, "f_Y"] & \\ X' \arrow[ur, hookleftarrow, "f_X"] \arrow[rr, "f"] & & Y' \end{tikzcd}$$

with f_X and f_Y atomic, we say that f is *compatible with extension* if f sends the unique color of $p(X') \setminus \mathrm{im}(p(f_X))$ to the unique color of $p(Y') \setminus \mathrm{im}(p(f_Y))$.

Remark 2.2.2. In [Lur17, Definition 3.3.2.3.], the notion of a *m-semi-inert morphism* is introduced, for $m \in \mathbb{N}$. In terms of this definition, a morphism f in $\mathcal{O}^{\otimes}$ is atomic if and only if it is 1-semi-inert but not 0-semi-inert.

Definition 2.2.3 (∞-category of extensions)**.** Let $\sigma\colon \Delta^n \to \mathcal{O}^{\otimes}_{\mathrm{act}}$ be an n-simplex corresponding to a sequence of active morphisms $X_0 \xrightarrow{f_1} \dots \xrightarrow{f_n} X_n$. Given a downward-closed subset $S = \{0, \dots, r\} \subseteq [n]$, let $\mathrm{Ext}(\sigma, S)$ be the (non-full) subcategory of $\mathrm{Fun}(\Delta^n, \mathcal{O}^{\otimes})_{\sigma/}$ whose

- **objects** are diagrams $\Delta^1 \times \Delta^n \to \mathcal{O}^{\otimes}$ represented as

$$\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[d, hook, "g_0"] & \dots \arrow[r, "f_r"] & X_r \arrow[r] \arrow[d, hook, "g_r"] & X_{r+1} \arrow[r] \arrow[d, "g_{r+1}", "\wr"'] & \dots \arrow[r, "f_n"] & X_n \arrow[d, "g_n", "\wr"'] \\
X'_0 \arrow[r, "f'_1"] & \dots \arrow[r, "f'_r"] & X'_r \arrow[r] & X'_{r+1} \arrow[r] & \dots \arrow[r, "f'_n"] & X'_n
\end{tikzcd}$$

 satisfying the following conditions:

 (1) if $i \notin S$, then g_i is an equivalence,

 (2) if $i \in S$, then g_i is atomic,

 (3) if $i \in S \setminus \{0\}$, then f'_i is compatible with extension,

 (4) each map f'_i is active;

- **morphisms** are diagrams $\Delta^2 \times \Delta^n \to \mathcal{O}^{\otimes}$ represented as

$$\begin{tikzcd}
X_0 \arrow[r, "f_1"] \arrow[d, hook, "g_0"] \arrow[dd, bend right=60] & \dots \arrow[r, "f_n"] & X_n \arrow[d, "g_n"', "\wr"] \arrow[dd, bend left=60, "\wr"] \\
X'_0 \arrow[r, "f'_1"] \arrow[d, "h_0"] & \dots \arrow[r, "f'_n"] & X'_n \arrow[d, "h_n"'] \\
X''_0 \arrow[r, "f''_1"] & \dots \arrow[r, "f''_n"] & X''_n
\end{tikzcd}$$

 in which the morphisms $h_i\colon X'_i \to X''_i$ are compatible with extension for all $i \in S$.

Given an active morphism $\sigma\colon \Delta^1 \to \mathcal{O}^{\otimes}_{\mathrm{act}}$, we write $\mathrm{Ext}(\sigma)$ for $\mathrm{Ext}(\sigma, \{0\})$. We call $\mathrm{Ext}(\sigma)$ the ∞-*category of extensions* of σ. When the underlying ∞-category $\mathcal{O}$ of $\mathcal{O}^{\otimes}$ is an ∞-groupoid, $\mathrm{Ext}(\sigma)$ is a Kan complex and therefore refered to as the *space* of extensions of σ.

Example 2.2.4 (Description of $\mathrm{Ext}(\sigma)$ in the discrete case). Let $\mathcal{O}_{\Delta}$ be an operad in sets, $\mathcal{O}^{\otimes}$ its homotopy coherent nerve and $\sigma\colon \langle m \rangle \to \langle 1 \rangle$ an active morphism in $\mathcal{O}^{\otimes}$. Then the k-simplices of $\mathrm{Ext}(\sigma)$ are those functors between 1-categories

$[1] \times [1+k] \to \mathcal{O}_\Delta$ whose associated diagrams is of the form

$$\begin{tikzcd}
\langle m \rangle \arrow[r, "\sigma"] \arrow[d, hook, "\text{atomic}"] & \langle 1 \rangle \arrow[d, "\sim"] \\
\langle m+1 \rangle \arrow[r, "\text{act}"] & \langle 1 \rangle \\
\vdots & \vdots \\
\langle m+1 \rangle \arrow[r, "\text{act}"] \arrow[d] & \langle 1 \rangle \arrow[d, "\sim"] \\
\langle m+1 \rangle \arrow[r, "\text{act}"] & \langle 1 \rangle .
\end{tikzcd} \tag{2.1}$$

and such that all the left vertical morphism $\langle m+1 \rangle \to \langle m+1 \rangle$ are compatible with extensions.

Remark 2.2.5 (Difference with the existing definition). The previous definition is slightly different from the initial definition from [Lur17] in that we impose a condition on the morphisms in $\mathrm{Ext}(\sigma)$, rather than defining it as a *full* subcategory of $\mathrm{Fun}(\Delta^n, \mathcal{O}^\otimes)_{\sigma/}$. The reason for this choice is that the space defined in [Lur17, Definition 3.3.1.4.], that we shall denote $\mathrm{Ext}^{\mathrm{HA}}(\sigma)$ here, does not have the expected homotopy type. To see this, consider the example of the commutative ∞-operad $\mathcal{O}^\otimes = \mathrm{Comm}^\otimes$ and $\sigma \colon \langle m \rangle \to \langle 1 \rangle$ be an active map in $\mathrm{Comm}^\otimes$. As described in [Lur17, Example 3.3.1.12], the space of extensions of σ is supposed to be the singleton set $\langle 1 \rangle^\circ$, viewed as a discrete space. However, the space $\mathrm{Ext}^{\mathrm{HA}}(\sigma)$ is not discrete. Indeed, consider the object $\alpha \in \mathrm{Ext}^{\mathrm{HA}}(\sigma)$ given by the following diagram

$$\begin{tikzcd}
\langle m \rangle \arrow[r, "\sigma"] \arrow[d, hook, "i"] & \langle 1 \rangle \arrow[d, "\mathrm{id}"] \\
\langle m+1 \rangle \arrow[r, "!"] & \langle 1 \rangle
\end{tikzcd}$$

where m is a positive integer, $! \colon \langle m+1 \rangle \to \langle 1 \rangle$ is the unique active map and i the canonical inclusion. We claim that $\pi_1(\mathrm{Ext}^{\mathrm{HA}}(\sigma), \alpha)$ is not trivial. Let $\mu \colon \langle m+1 \rangle \to \langle m+1 \rangle$ be the morphism in $\mathrm{Comm}^\otimes_{\mathrm{act}}$ that restricts to the atomic morphism i on $\langle m \rangle$ and sends the remaining color $m+1$ to 1. Then the diagram

$$\begin{tikzcd}
\langle m \rangle \arrow[r, "\sigma"] \arrow[d, hook, "i"] \arrow[dd, bend right=60, hook, "i"'] & \langle 1 \rangle \arrow[d, "\mathrm{id}"] \arrow[dd, bend left=60, "\mathrm{id}"] \\
\langle m+1 \rangle \arrow[r, "!"] \arrow[d, "\mu"] & \langle 1 \rangle \arrow[d, "\mathrm{id}"] \\
\langle m+1 \rangle \arrow[r, "!"] & \langle 1 \rangle
\end{tikzcd} \tag{2.2}$$

defines a morphism $\gamma \colon \alpha \to \alpha$ in $\mathrm{Ext}^{\mathrm{HA}}(\sigma)$ with the property that $[\gamma] \neq [\mathrm{id}_\alpha]$ in $\pi_1(\mathrm{Ext}^{\mathrm{HA}}(\sigma), \alpha)$. Indeed, a homotopy between γ and id_α would give a retraction ρ of μ, which can't be.

Note that diagram (2.2) does not define a morphism in $\mathrm{Ext}(\sigma)$ since μ is not compatible with extensions. We will see that definition 2.2.3 yields the expected homotopy type for the spaces of extensions: this is the content of theorem C.

Definition 2.2.6 ([Lur17, Definition 3.3.1.9])**.** An ∞-operad $\mathcal{O}^{\otimes}$ is *coherent* if it satisfies the following conditions:

(a) it is unital,

(b) its underlying ∞-category $\mathcal{O}$ is an ∞-groupoid,

(c) for every degenerate 3-simplex σ

$$\begin{tikzcd} & Y \arrow[rr, "g"] \arrow[dr, "\mathrm{id}_Y"] & & Z \\ X \arrow[ur, "f"] \arrow[rr, "f"] & & Y \arrow[ur, "g"] & \end{tikzcd}$$

in $\mathcal{O}^{\otimes}_{\mathrm{act}}$, the commutative diagram

$$\begin{tikzcd} \mathrm{Ext}(\sigma, \{0,1\}) \arrow[r] \arrow[d] & \mathrm{Ext}(\sigma|_{\Delta^{\{0,1,3\}}}, \{0,1\}) \arrow[d] \\ \mathrm{Ext}(\sigma|_{\Delta^{\{0,2,3\}}}, \{0\}) \arrow[r] & \mathrm{Ext}(\sigma|_{\Delta^{\{0,3\}}}, \{0\}) \end{tikzcd} \tag{2.3}$$

is a homotopy cocartesian square of Kan complexes.

Remark 2.2.7. Let σ and $S = \{0, \dots, r\}$ be as in definition 2.2.3 and suppose that $\mathcal{O}$ is an ∞-groupoid. As mentionned before, the simplicial set $\mathrm{Ext}(\sigma)$ is a Kan complex. By remark [Lur17, Remark 3.3.1.6.], if $r < [n]$, there is a canonical map $\mathrm{Ext}(\sigma, S) \to \mathrm{Ext}(f_{r+1})$ which is trivial Kan fibration. Using these equivalences, we may rewrite the commutative square (2.3) as

$$\begin{tikzcd} \mathrm{Ext}(\mathrm{id}_Y) \arrow[r] \arrow[d] & \mathrm{Ext}(g) \arrow[d] \\ \mathrm{Ext}(f) \arrow[r] & \mathrm{Ext}(g \circ f). \end{tikzcd} \tag{2.4}$$

Note that the previous square is only well-defined in the homotopy category of spaces.

2.3 Symmetric monoidal envelope and its twisted arrows

Recall the construction of the symmetric monoidal envelope $\mathrm{Env}\colon \mathsf{Op}_\infty \to \mathsf{Cat}_\infty^{\otimes}$, which is left adjoint to the forgetful functor from symmetric monoidal ∞-categories to ∞-operads [Lur17, Section 2.2.4]. This left adjoint sends an ∞-operad $\mathcal{P}^{\otimes}$ to the ∞-category

$$\mathrm{Env}(\mathcal{P})^{\otimes} := \mathcal{P}^{\otimes} \times_{\mathrm{Fun}(\{0\},\mathbb{F}_*)} \mathrm{Fun}^{\mathrm{act}}(\Delta^1, \mathbb{F}_*),$$

where the superscript act indicates the full subcategory of $\mathrm{Fun}(\Delta^1, \mathbb{F}_*)$ whose objects are active morphisms in $\mathbb{F}_*$. This ∞-category inherits a symmetric monoidal structure via the functor $p_1\colon \mathrm{Env}(\mathcal{O})^{\otimes} \to \mathbb{F}_*$ given by evaluation at $1 \in \Delta^1$. Note that the underlying ∞-category $\mathrm{Env}(\mathcal{O})$ of $\mathrm{Env}(\mathcal{O})^{\otimes}$ can be identified with the wide subcategory $\mathcal{O}^{\otimes}_{\mathrm{act}}$ of $\mathcal{O}^{\otimes}$ consisting of all objects and only active maps between them. As explained in 2.1, the ∞-category of twisted arrows $\mathrm{Tw}\,(\mathrm{Env}(\mathcal{O}))$ inherits a symmetric monoidal structure from that of the monoidal envelope, also denoted $p_1\colon \mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes} \to \mathbb{F}_*$.

For later purposes, we let $p_0\colon \mathrm{Env}(\mathcal{O})^{\otimes} \to \mathbb{F}_*$ denote the functor given by evaluation at 0.

Notation 2.3.1. For simplicity, we will write $\mathcal{E}$ for $\mathrm{Env}(\mathcal{O})^{\otimes}$ and $\mathcal{T}$ for $\mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes}$.

Let us unravel the definitions of $\mathcal{E}$ and $\mathcal{T}$.

- An object in $\mathcal{E}_{\langle n\rangle}$ is given by an object $X \in \mathcal{O}^{\otimes}_{\langle k\rangle}$ together with an active map $\langle k\rangle \to \langle n\rangle$ in $\mathbb{F}_*$. In terms of the projection functors p_0 and p_1, we have that $p_0(X, \langle k\rangle \to \langle n\rangle) = \langle k\rangle$ and $p_1(X, \langle k\rangle \to \langle n\rangle) = \langle n\rangle$. Thus, we may think of the object $(X, \langle k\rangle \to \langle n\rangle)$ in $\mathcal{E}$ as a list of n objects $(X_1, \ldots, X_n)$ in $\mathcal{O}^{\otimes}$, with total arity $\oplus_{i=1}^{n} p_0(X_i) \cong \langle k\rangle$.

- A morphism f in $\mathcal{E}$ from $(X, p_0(X) \to \langle n\rangle)$ to $(Y, p_0(Y) \to \langle m\rangle)$ is a morphism $X \to Y$ in $\mathcal{O}^{\otimes}$ together with a commutative diagram

$$\begin{tikzcd} p_0(X) \arrow[r] \arrow[d] & p_0(Y) \arrow[d] \\ \langle n\rangle \arrow[r, "\alpha"] & \langle m\rangle. \end{tikzcd}$$

  In the case where α is active, the morphism f is p_1-cocartesian if and only if $X \to Y$ is an equivalence, by [Lur17, Lemma 2.2.4.15.]

- An object of $\mathcal{T}_{\langle n\rangle}$ is given by an active map $g\colon X \to Y$ in $\mathcal{O}^{\otimes}$ together with a commutative triangle

$$\begin{tikzcd} p_0(X) \arrow[rr, "p_0(g)"] \arrow[dr, "\mathrm{act}"'] & & p_0(Y) \arrow[dl, "\mathrm{act}"] \\ & \langle n\rangle & \end{tikzcd}$$

  of active maps in $\mathbb{F}_*$. A morphism in $\mathcal{T}$ from the previous object to $(g'\colon X' \to Y', p_0(Y') \to \langle m\rangle)$ is given as a pair of commutative diagrams

$$\begin{tikzcd} X \arrow[r] \arrow[d, "g"'] & X' \arrow[d, "g'"] \\ Y & Y' \arrow[l] \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} p_0(X) \arrow[r] \arrow[d, "p_0(g)"'] & p_0(X') \arrow[d, "p_0(g')"] \\ p_0(Y) \arrow[d, "\mathrm{act}"] & p_0(Y') \arrow[l] \arrow[d, "\mathrm{act}"] \\ \langle n\rangle \arrow[r, "\alpha"] & \langle m\rangle \end{tikzcd} \tag{2.5}$$

respectively in $\mathcal{O}^{\otimes}$ and $\mathbb{F}_*$. If one interprets the objects in $\mathcal{E}$ as lists of objects of $\mathcal{O}^{\otimes}$, then the equivalence $\mathcal{T}_{\langle m\rangle} \simeq \mathrm{Tw}(\mathcal{O}^{\otimes}_{\mathrm{act}})^m$ allows to view the object $(g\colon X \to Y, p_0(Y) \to \langle n\rangle)$ in $\mathcal{T}_{\langle n\rangle}$ as a list of n active morphisms $(X_1 \to Y_1, \dots, X_n \to Y_n)$ in $\mathcal{O}^{\otimes}$.

- A morphism in $\mathcal{T}$ between two objects $(\sigma_1\colon X_1 \to Y_1, \dots, \sigma_n\colon X_n \to Y_n)$ and $(\sigma'_1\colon X'_1 \to Y'_1, \dots, \sigma'_m\colon X'_m \to Y'_m)$ then corresponds to a morphism $\alpha\colon \langle n\rangle \to \langle m\rangle$ in $\mathbb{F}_*$ together with a commutative diagram in $\mathcal{O}^{\otimes}$ of the form

$$\begin{tikzcd} \bigoplus_{i\in\alpha^{-1}(j)} X_i \arrow[r] \arrow[d, "\oplus_i\sigma_i"', "\mathrm{act}"] & X'_j \arrow[d, "\mathrm{act}"] \\ \bigoplus_{i\in\alpha^{-1}(j)} Y_i & Y'_j \arrow[l] \end{tikzcd}$$

for each $j \in \langle m\rangle^{\circ}$.

Remark 2.3.2. Consider the morphism in $\mathcal{T}$ given by diagrams (2.5) and assume that α is active. Then this morphism is p_1-cocartesian if and only if both maps $X \to X'$ and $Y' \to Y$ are equivalences.

2.4 Construction of the brane fibration

To prove theorem A, we will follow the strategy developed by Mann–Robalo in [MR18, Section 2.1]. Let us recall their approach.

Mann–Robalo's strategy

First, note that the datum of a map of ∞-operads $\mathcal{O}^{\otimes} \to \mathrm{Cospan}(\mathcal{S})^{\otimes}$ is equivalent to that of a map of symmetric monoidal functors $\mathcal{E} \to \mathrm{Cospan}(\mathcal{S})^{\otimes}$. By the universal property of spans (proposition 2.1.3), this datum is equivalently that of a symmetric monoidal functor $\mathcal{T} \to (\mathcal{S}^{\mathrm{op}})^{\amalg}$ satisfying the pullback condition. By [Lur17, Proposition 2.4.1.7.], since the monoidal structure $(\mathcal{S}^{\mathrm{op}})^{\amalg}$ on $\mathcal{S}^{\mathrm{op}}$ is cartesian, this is the same as providing a weak cartesian structure $\mathcal{T} \to \mathcal{S}^{\mathrm{op}}$ satisfying the pullback condition. Using the Grothendieck construction, it will suffice to construct a right fibration $\pi\colon \mathcal{BO} \to \mathcal{T}$ whose classifying functor $F_\pi\colon \mathcal{T} \to \mathcal{S}^{\mathrm{op}}$ satisfies the conditions described above.

The rest of this section is devoted to the construction of this fibration π.

Definition 2.4.1 (The brane fibration, following [MR18])**.** Define $\mathcal{BO}$ as the subsimplicial set of $\mathrm{Fun}(\Delta^1, \mathcal{T})$ whose

- **objects** are twisted morphisms $\sigma \rightsquigarrow \sigma^+$ such that
  - the projection $p_1(\sigma \rightsquigarrow \sigma^+)$ in $\mathbb{F}_*$ is the unique active map $p_1(\sigma) \to \langle 1\rangle$;

– in the corresponding 3-simplex in $\mathcal{O}^{\otimes}$

$$\begin{tikzcd} S_0 \arrow[r, hook, "\sigma_0"] \arrow[d, "\sigma"'] & S_0^+ \arrow[d, "\sigma^+"] \\ S_1 & S_1^+ \arrow[l, "\sim"', "\sigma_1"] \end{tikzcd} \tag{2.6}$$

the map σ_0 is atomic and σ_1 is an equivalence;

- **morphisms** from $\sigma \rightsquigarrow \sigma^+$ to $\tau \rightsquigarrow \tau^+$ are the morphisms f in $\mathrm{Fun}(\Delta^1, \mathcal{T})$ such that

  – the projection $p_1\left(\begin{tikzcd} \sigma \arrow[r, squiggly] \arrow[d, squiggly] & \sigma^+ \arrow[d, squiggly] \\ \tau \arrow[r, squiggly] & \tau^+ \end{tikzcd}\right)$ in $\mathbb{F}_*$ is the diagram

  $$\begin{tikzcd} p_1(\sigma) \arrow[r, "\mathrm{act}"] \arrow[d] & \langle 1 \rangle \arrow[d, "\mathrm{id}"] \\ p_1(\tau) \arrow[r, "\mathrm{act}"] & \langle 1 \rangle, \end{tikzcd}$$

  – in the induced square

  $$\begin{tikzcd} S_0 \arrow[r, hook, "\sigma_0"] \arrow[d, "f_0"'] & S_0^+ \arrow[d, "f_0^+"] \\ T_0 \arrow[r, hook, "\tau_0"] & T_0^+, \end{tikzcd} \tag{2.7}$$

  the morphism f_0^+ is *compatible with extension*, in the sense that $p_0(f_0^+)$ is of the form $\langle s+1 \rangle \to \langle t+1 \rangle$, sending the singleton $\langle s+1 \rangle \setminus \mathrm{im}(p_0(\sigma_0))$ to the singleton $\langle t+1 \rangle \setminus \mathrm{im}(p_0(\tau_0))$.

Let $\pi\colon \mathcal{BO} \to \mathcal{T}$ be the composite of ev_0 with the inclusion $\mathcal{BO} \subset \mathrm{Fun}(\Delta^1, \mathcal{T})$.

Remark 2.4.2. The following properties will be useful throughout this paper.

- Since equivalences and atomic maps are active, the diagram (2.6) is in fact in $\mathcal{O}^{\otimes}_{\mathrm{act}}$.
- The image of $\mathcal{BO}$ under p_1 is constant along the fibers of π, in the sense that there is a commutative diagram

$$\begin{tikzcd} \mathcal{BO} \arrow[r, hook] \arrow[d, "\pi"'] & \mathrm{Fun}(\Delta^1, \mathcal{T}) \arrow[dd, "p_1 \circ -"] \\ \mathcal{T} \arrow[d, "p_1"'] & \\ \mathbb{F}_* \arrow[r, "\varepsilon"] & \mathrm{Fun}(\Delta^1, \mathbb{F}_*). \end{tikzcd}$$

Here ε is the unique functor that sends $\langle n\rangle$ to the unique active morphism $\langle n\rangle \to \langle 1\rangle$ and such that $\mathrm{ev}_0 \circ \varepsilon = \mathrm{id}_{\mathbb{F}_*}$ and $\mathrm{ev}_1 \circ \varepsilon = \mathrm{const}_{\langle 1\rangle}$. Thanks to this observation, we will often leave implicit the description of the projection under p_1 of various constructions.

Let us mention the following general facts about $\mathcal{BO}$.

Lemma 2.4.3. *The inclusion $\mathcal{BO} \subset \mathrm{Fun}(\Delta^1, \mathcal{T})$ is a conservative isofibration. In particular, $\mathcal{BO}$ is an ∞-category.*

Proof. We have to show that $\mathcal{BO}$ is a *replete subcategory* (in the sense of [Lur22, Definition 01CF] and [Lur22, Example 01EX]) of $\mathrm{Fun}(\Delta^1, \mathcal{T})$. First, we verify the conditions of [Lur22, Corollary 01CR] to prove that $\mathcal{BO}$ is a subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{T})$. As the condition of compatibility with extension of 2.4.1 only depends of the image of the morphisms in the 1-category $\mathbb{F}_*$, one easily verifies that the set of morphisms in $\mathcal{BO}$ contain all identities of objects in $\mathcal{BO}$ is closed under homotopy and composition, as desired.
Next, we turn to the proof that $\mathcal{BO}$ is replete. Let $f^+ \colon \sigma^+ \to \tau^+$ be an equivalence in $\mathrm{Fun}(\Delta^1, \mathcal{T})$ with $\sigma^+ \in \mathcal{BO}$. We have to show that both τ^+ and f^+ belong to $\mathcal{BO}$. Since the canonical functor $\mathrm{Tw}(\mathcal{O}^\otimes) \to \mathcal{O}^\otimes \times (\mathcal{O}^\otimes)^{\mathrm{op}}$ is conservative (being a right fibration), we deduce that in the diagram induced by f^+ in $\mathcal{O}^\otimes$, all four morphisms $f_0 \colon S_0 \to T_0$, $f_0^+ \colon S_0^+ \to T_0^+$, $f_1 \colon T_1 \to S_1$, $f_1^+ \colon T_1^+ \to S_1^+$ are equivalences. From this and the commutativity of the square (2.7), one obtains that $\tau_0 \colon T_0 \to T_0^+$ is semi-inert, lies over an injection $\langle t\rangle \hookrightarrow \langle t+1\rangle$ and that f_0^+ is compatible with extension. Similarly, the morphisms σ_1, f_1 and f_1^+ are equivalences, therefore so must be τ_1. This concludes the proof. □

Lemma 2.4.4. *Assume that the underlying ∞-category $\mathcal{O}$ of $\mathcal{O}^\otimes$ is an ∞-groupoid and let $\sigma \in \mathcal{T}$. Then the fiber $\mathcal{BO}_\sigma$ of π at σ is a Kan complex.*

Proof. By the previous lemma, the inclusion $\mathcal{BO} \subset \mathrm{Fun}(\Delta^1, \mathcal{T})$ is a conservative isofibration. So is the map $\mathrm{ev}_0 \colon \mathrm{Fun}(\Delta^1, \mathcal{T}) \to \mathcal{T}$, hence π is an isofibration. To prove the result, it now suffices to show that π is conservative. Consider a morphism $f \colon \sigma^+ \to \tau^+$ in $\mathcal{BO}$ whose image $\pi(f)$ in $\mathcal{T}$ is an equivalence. The data of f is that of a diagram of the following form:

$$
\begin{tikzcd}
& S_0^+ \arrow[rr, "f_0^+"] \arrow[dd, "\sigma_0^+" near end] & & T_0^+ \arrow[dd, "\tau^+"] \\
S_0 \arrow[ur, "\sigma_0"] \arrow[rr, "f_0" near end] \arrow[dd, "\sigma"] & & T_0 \arrow[ur, "\tau_0"] \arrow[dd, "\tau" near start] & \\
& S_1^+ \arrow[dl, "\sigma_1"] & & T_1^+. \arrow[ll, "f_1^+" near start] \arrow[dl, "\tau_1"] \\
S_1 & & T_1 \arrow[ll, "f_1"] &
\end{tikzcd}
\tag{2.8}
$$

Using that the fibration $\mathcal{T} \to \mathcal{E} \times \mathcal{E}^{\mathrm{op}}$ is conservative, we deduce that f_0 and f_1 are equivalences. By definition of the objects in $\mathcal{BO}$, the maps σ_1 and τ_1 are also equivalences, therefore so is f_1^+. Finally, we claim that f_0^+ is an equivalence. To see this, write f_0^+ as the sum $f_0 \oplus f_0^+|_+$, where $f_0^+|_+ \colon S_0^+ \setminus S_0 \to T_0^+ \setminus T_0$ is the restriction of f_0^+ to the new color. Since $f_0^+|_+$ is a map in $\mathcal{O}$, which by assumption is an ∞-groupoid, it is an equivalence; therefore so is f. □

2.5 Proof of theorem A

One of the key steps in the proof of theorem A is the following result, whose proof is given in chapter 3.

Theorem 2.5.1. *Let $\mathcal{O}^\otimes$ be a unital ∞-operad. Then the functor $\pi \colon \mathcal{BO} \to \mathcal{T}$ is a cartesian fibration.*

Assuming theorems 2.5.1 and B, whose proofs will be given in chapters 3 and 4, we can prove theorem A.

Proof of theorem A. By Mann–Robalo's argument (as described in section 2.4), in order to prove the theorem it suffices to construct a right fibration over $\mathcal{T}$, with fibers equivalent to spaces of extensions and whose associated functor $\mathcal{T} \to \mathcal{S}^{\mathrm{op}}$ is a weak cartesian structure and satisfies the pullback condition of proposition 2.1.3.

Theorem 2.5.1 ensures that π is a cartesian fibration. By lemma 2.4.4, its fibers are Kan complexes, hence π is a right fibration. Moreover, theorem B identifies the fiber $\mathcal{BO}_\sigma$ over an object $\sigma \in \mathcal{T}$ as its space of extensions $\mathrm{Ext}(\sigma)$. Therefore, it remains to show that the functor $F_\pi \colon \mathcal{T} \to \mathcal{S}^{\mathrm{op}}$ classifying the right fibration π is a weak cartesian structure and satisfies the pullback condition. The latter is exactly the condition that the ∞-operad $\mathcal{O}^\otimes$ is coherent, using the equivalence $\mathcal{BO}_\sigma \simeq \mathrm{Ext}(\sigma)$. For the weak cartesian condition, let σ in $\mathcal{T}$ be decomposed as a sum $\sigma \simeq \oplus_{i=1}^n \sigma_i$ of objects in $\mathcal{T}_{\langle 1 \rangle} \simeq \mathrm{Tw}(\mathcal{O}^\otimes_{\mathrm{act}})$. Since p_1 is constant along fibers of π (in the sense of remark 2.4.2), the fiber $\mathcal{BO}_\sigma$ decomposes as a disjoint union of the spaces $\mathcal{BO}_{\sigma_i}$, so that the natural map $F_\pi(\sigma) \xleftarrow{\sim} \coprod_{i=1}^n F_\pi(\sigma_i)$ in $\mathcal{S}$ is an equivalence. This shows that F_π is a lax cartesian structure. To verify that it is in fact a weak cartesian structure, let $f \colon \sigma \to \sigma'$ in $\mathcal{T}$ be a p_1-cocartesian lift of the unique active morphism $\langle n \rangle \to \langle 1 \rangle$ in $\mathbb{F}_*$. By remark 2.3.2, this implies that the two maps $\mathrm{source}(\sigma) \to \mathrm{source}(\sigma')$ and $\mathrm{target}(\sigma) \leftarrow \mathrm{target}(\sigma')$ are equivalences, which in turn ensures that $F_\pi(f)$ is an equivalence, as desired. □

2.6 Generalized version of theorem A

The brane action given by theorem A can be generalized to the setting where $\mathcal{O}^\otimes$ is a unital ∞-operad, without assuming that its underlying ∞-category $\mathcal{O}$ is an ∞-groupoid (condition (b) in the definition of coherence given in 2.2.6).

To make this claim precise, let us say that an ∞-operad $\mathcal{O}^\otimes$ is *categorically coherent* if it is unital and satisfies the variant (c') of condition (c) in definition 2.2.6 in which one requires diagram (2.3) to be a *categorical* pushout square of

∞-categories (instead of a homotopy pushout square). Note that if $\mathcal{O}^{\otimes}$ is unital with $\mathcal{O}^{\otimes}$ an ∞-groupoid, i.e. $\mathcal{O}^{\otimes}$ satisfies conditions (a) and (b) from definition 2.2.6, then conditions (c) and (c') actually coincide, since for Kan complexes, homotopy pushout squares are automatically categorical pushout squares. As a consequence, coherent ∞-operads are categorically coherent. The generalized version of theorem A writes as follows.

Theorem A'. *Let $\mathcal{O}^{\otimes}$ be a categorically coherent ∞-operad. Then the collection of ∞-categories $\{\mathrm{Ext}(\mathrm{id}_c)\}_{c\in\mathcal{O}}$ carries a canonical $\mathcal{O}$-algebra structure in* $\mathrm{Cospan}(\mathsf{Cat}_{\infty})$*, which recovers that of theorem A when $\mathcal{O}$ is an ∞-groupoid.*

The proof of theorem A' is almost the same as the one given above for theorem A, only slightly simpler. Indeed, most of the arguments, including the use of theorems 2.5.1 and B, do not use the assumption that $\mathcal{O}$ is an ∞-groupoid. The only difference is that in the situation of theorem A', π is merely a cartesian fibration (as opposed to a right fibration) and therefore its classifying functor is of the form $\mathcal{T} \to \mathsf{Cat}_{\infty}^{\mathrm{op}}$.

Following [Toë13], one may go one step further in generality by dropping the assumption that $\mathcal{O}^{\otimes}$ is coherent, that is assuming only that $\mathcal{O}^{\otimes}$ is a unital ∞-operad. In this case the brane action merely gives a *lax* algebra structure on the ∞-category $\mathrm{Ext}(\sigma)$ in cospans of ∞-categories, which is an genuine algebra structure precisely when $\mathcal{O}^{\otimes}$ is coherent (in the previous generalized sense). We refer to Kern's thesis [Ker21] for more details on this lax structure.

Chapter 3

Cartesianity of the brane fibration

This chapter is devoted to the proof of theorem 2.5.1, asserting that the brane fibration $\pi\colon \mathcal{BO} \to \mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes}$ of definition 2.4.1 is indeed a cartesian fibration. We will define particular lifts of edges along π and then show that these are cartesian arrows in $\mathcal{BO}$ in the rest of the chapter. Note that cartesianity of this fibration is the property ensures the existence of all the homotopical coherences involved in the definition of the $\mathcal{O}$-algebra in $\mathrm{Cospan}(\mathcal{S})$ given by the brane action.

Contents

3.1 Construction of cartesian lifts

Let $f\colon \sigma \rightsquigarrow \tau$ be a morphism in $\mathcal{T}$ and let $e_\tau\colon \tau \rightsquigarrow \tau^+$ be in the fiber $\mathcal{BO}_\tau$. We will construct a cartesian edge $f^+\colon \sigma^+ \rightsquigarrow \tau^+$ lying π-above f.

$$\begin{tikzcd}
\sigma^+ \arrow[r, dashed, "f^+"] & \tau^+ & & \mathcal{BO} \arrow[d, "\pi"] \\
\sigma \arrow[u, dashed] \arrow[r, "f"] & \tau \arrow[u, "e_\tau"'] & & \mathcal{T}
\end{tikzcd}$$

Unraveling the definition of $\mathcal{BO}$, we are given a diagram of the form

$$\begin{tikzcd}
& & T_0^+ \arrow[dd, "\tau^+"] \\
S_0 \arrow[r, "f_0"] \arrow[dd, "\sigma"] & T_0 \arrow[ur, hook, "\tau_0"] \arrow[dd, "\tau"] & \\
& & T_1^+ \arrow[dl, "\tau_1"] \\
S_1 & T_1 \arrow[l, "f_1"] &
\end{tikzcd} \tag{3.1}$$

in $\mathcal{E}$ and want to extend it to one of the shape

$$\begin{tikzcd}
& S_0^+ \arrow[rr, dashed, "f_0^+"] \arrow[dd, dashed] & & T_0^+ \arrow[dd, "\tau^+"] \\
S_0 \arrow[ur, hook, dashed, "\sigma_0"] \arrow[rr, "f_0"] \arrow[dd, "\sigma"] & & T_0 \arrow[ur, hook, "\tau_0"] \arrow[dd, "\tau"] & \\
& S_1^+ \arrow[dl, dashed, "\sigma_1"] & & T_1^+ \arrow[ll, dashed, "f_1^+"] \arrow[dl, "\tau_1"] \\
S_1 & & T_1 \arrow[ll, "f_1"] &
\end{tikzcd} \tag{3.2}$$

so that the resulting morphism $f^+ \colon \sigma^+ \rightsquigarrow \tau^+$ is in $\mathcal{BO}$. We proceed in several steps, depicted in figure 3.1.

Step 1. Pick a representative $\tilde{f}$ for the composite $e_\tau \circ f$ in $\mathcal{T}$. In particular, this yields a 3-simplex $S_0T_0^+T_1^+S_1$ and a 5-simplex $S_0T_0T_0^+T_1^+T_1S_1$ extending diagram (3.1).

Step 2. Define the object S_1^+ as S_1 and the morphism $\sigma_1 \colon S_1^+ \to S_1$ as the identity. Since σ_1 is an equivalence, by using Joyal's lifting theorem [Lur22, Theorem 019F] and several horn fillers, we can extend the 3-simplex $S_0T_0^+T_1^+S_1$ to a 4-simplex $S_0T_0^+T_1^+S_1^+S_1$.

Step 3. We now turn to the key step, namely the construction of the triangle $S_0S_0^+T_0^+$. Decompose T_0^+ as a sum of colors

$$T_0^+ = \oplus_{i \in p_0(T_0)} C_i \oplus C^+$$

so that C^+ is the color lying above the element $p_0(T_0^+) \setminus \operatorname{im}(p_0(\tau_0))$. Since $\mathcal{O}^\otimes$ is unital, there exists an essentially unique morphism ι_{C^+} from the zero object of $\mathcal{O}^\otimes$ to C^+. Define S_0^+ as the sum $S_0 \oplus C^+$ and σ_0 as $\mathrm{id}_{S_0} \oplus \iota_{C^+}$, which is clearly an atomic morphism.

It remains to construct f_0^+. Note that $p_0(f_0^+)$ is required to coincide with the unique morphism $h \colon p_0(S_0^+) \to p_0(T_0^+)$ that restricts to $p_0(\tilde{f}_0)$ on $p_0(S_0)$ and preserves $p_0(C^+)$. Consider the ∞-category

$$\mathcal{M} = (\mathcal{O}^\otimes)^{\Delta^2} \times_{(\mathcal{O}^\otimes)^{\Delta^{\{2\}}}} \{T_0^+\} \times_{(\mathcal{O}^\otimes)^{\Delta^{01}}} \{\sigma_0\} \times_{\mathbb{F}_*^{\Lambda_2^2}} \{(p_0(\tilde{f}_0), h)\}$$

consisting of all diagrams of the form

$$\begin{tikzcd} & S_0^+ \arrow[dr, dashed, "a"] & \\ S_0 \arrow[ur, "\sigma_0"] \arrow[rr, dashed, "b"] & & T_0^+ \end{tikzcd} \tag{3.3}$$

satisfying $p_0(a) = h$ and $p_0(b) = p_0(\tilde{f}_0)$. The inclusion $\Delta^{02} \hookrightarrow \Delta^2$ yields a morphism

$$\mathcal{M} \longrightarrow \mathrm{Map}_{\mathcal{O}^\otimes}^{p_0(\tilde{f}_0)}(S_0, T_0^+) \xrightarrow{\sim} \prod_{i=1}^{s} \mathrm{Map}_{\mathcal{O}^\otimes}^{\rho^i \circ h}(S_0, C_i). \tag{3.4}$$

On the other hand, from the inner anodyne inclusion $\Lambda_1^2 \hookrightarrow \Delta^2$ and the definition of ∞-operads, we get the following sequence of equivalences

$$\begin{aligned} \mathcal{M} &\xrightarrow{\sim} \mathrm{Map}_{\mathcal{O}^\otimes}^{h}(S_0^+, T_0^+) \\ &\xrightarrow{\sim} \prod_{i=1}^{s} \mathrm{Map}_{\mathcal{O}^\otimes}^{\rho^i \circ h}(S_0^+, C_i) \times \mathrm{Map}_{\mathcal{O}^\otimes}^{\rho^{n+1} \circ h}(S_0^+, C^+) \\ &\xrightarrow{\sim} \prod_{i=1}^{s} \mathrm{Map}_{\mathcal{O}^\otimes}^{\rho^i \circ h}(S_0, C_i) \times \mathrm{Map}_{\mathcal{O}}(C^+, C^+). \end{aligned}$$

Composing those equivalences with the projection $\mathrm{Map}_{\mathcal{O}}(C^+, C^+) \to *$ recovers exactly the morphism (3.4). Therefore we see that the ∞-category of diagrams of the form (3.3) satisfying that $b = \tilde{f}_0$, which we identify with the fiber of the morphism (3.4) at $\tilde{f}_0$, is equivalent to $\mathrm{Map}_{\mathcal{O}}(C^+, C^+)$.

To define f_0^+ and a corresponding 2-simplex of diagram (3.3), it then suffices to specify any object in this ∞-groupoid $\mathrm{Map}_{\mathcal{O}}(C^+, C^+)$.

Step 4. At that point, we have extended the 3-simplex $S_0 T_0^+ T_1^+ S_1$ to a diagram of shape

$$\Delta^{S_0 T_0^+ T_1^+ S_1^+ S_1} \cup_{\Delta^{S_0 T_0^+}} \Delta^{S_0 S_0^+ T_0^+}.$$

A simple computation shows that the inclusion of the latter simplicial set into $\Delta^{S_0 S_0^+ T_0^+ T_1^+ S_1^+ S_1}$ is inner anodyne; this allows us to choose an extension of this diagram to a 5-simplex $S_0 S_0^+ T_0^+ T_1^+ S_1^+ S_1$.

This completes the construction of an edge $f^+ \colon \sigma^+ \rightsquigarrow \tau^+$ lifting f. The bulk of the proof of theorem 2.5.1 consists in proving that f^+ is cartesian.

3.2 Outline of the proof of cartesianity

Given a morphism $f \colon \sigma \to \tau$ in $\mathcal{T}$ and an object τ^+ in $\mathcal{BO}$, we have constructed a particular edge $f^+ \colon \sigma^+ \to \tau^+$ lying over f, which can be interpreted as an object in the ∞-category $\mathcal{BO}_{/\tau^+} \times_{\mathcal{T}_{/\tau}} \mathcal{T}_{/f}$. The purpose of this section is to give an overview of the proof that f^+ is cartesian. The details will be dealt with in the rest of the chapter.

$$
\begin{tikzcd}
 & & T_0^+ \arrow[dd] \\
S_0 \arrow[urr, dashed, "\tilde{f}_0"] \arrow[r] \arrow[dd] & T_0 \arrow[ur, hook] \arrow[dd] & \\
 & & T_1^+ \arrow[dll, dashed] \arrow[dl] \\
S_1 & T_1 \arrow[l] &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & T_0^+ \arrow[dl, dashed] \arrow[dd] \\
S_0 \arrow[urr] \arrow[dr, dashed] \arrow[dd] & & \\
 & S_1^+ \arrow[dl, equal] & T_1^+ \arrow[l, dashed] \arrow[dll] \\
S_1 & &
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & S_0^+ \arrow[r, dashed, "f_0^+"] & T_0^+ \arrow[dl] \arrow[dd] \\
S_0 \arrow[ur, dashed] \arrow[urr] \arrow[dr] \arrow[dd] & & \\
 & S_1^+ \arrow[dl, equal] & T_1^+ \arrow[l] \arrow[dll] \\
S_1 & &
\end{tikzcd}
$$

Figure 3.1: Diagrams of steps 1, 2 and 3 of the construction of f^+. An arrow is dashed if it is added at the current step.

Notation 3.2.1. Throughout the proof, we will make use of the notation introduced in A.2.2. In other words, from now on, we fix an object $\nu^+ \in \mathcal{BO}$, write $\mathcal{D}_{\nu^+}$ for the ∞-category $\mathcal{BO}_{/\tau^+} \times_{\mathcal{T}_{/\tau}} \mathcal{T}_{/f} \times_{\mathcal{BO}} \{\nu^+\}$ and fix an object u in it. We then consider the associated space of lifts

$$\mathcal{L} = \mathcal{BO}_{/f^+} \times_{\mathcal{BO}} \{\nu^+\} \times_{\mathcal{D}_{\nu^+}} \{u\}.$$

More explicitly, the datum of the object $u \in \mathcal{D}_{\nu^+}$ is that of a triangle u_0 in $\mathcal{T}$ of the form

$$
\begin{tikzcd}
 & \sigma \arrow[dr, "f"] & \\
\nu \arrow[ur] \arrow[rr, "g"] & & \tau
\end{tikzcd}
\tag{3.5}
$$

together with a morphism $g^+ \colon \nu^+ \to \tau^+$ in $\mathcal{BO}$ lying π-above g. An object in $\mathcal{L}$ is a lift of u, that is the datum of a triangle

$$
\begin{tikzcd}
 & \sigma^+ \arrow[dr, "f^+"] & \\
\nu^+ \arrow[ur] \arrow[rr, "g^+"] & & \tau^+
\end{tikzcd}
\tag{3.6}
$$

in $\mathcal{BO}$ that lies π-above the triangle u_0 depicted in (3.5). In the diagrams parametrized by this ∞-groupoid $\mathcal{L}$, only the morphism $\nu^+ \to \sigma^+$ and the 2-simplex filling triangle (3.6) are allowed to vary.

By lemma A.2.3, proving that f^+ is π-cartesian amounts to showing the following result.

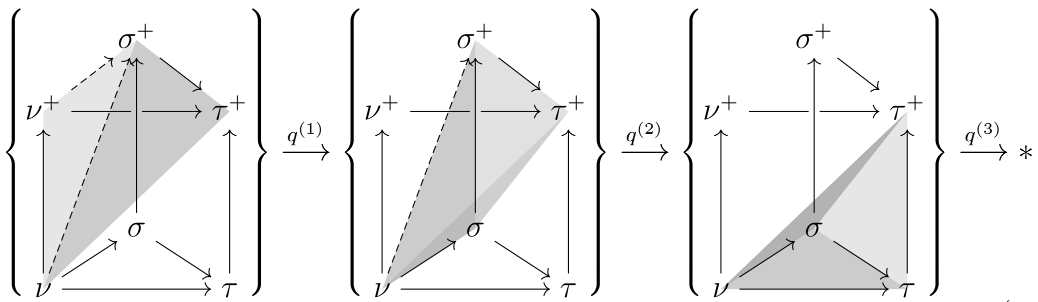
Proposition 3.2.2. *The space of lifts $\mathcal{L}$ is contractible.*

The rest of this chapter is devoted to the proof of proposition 3.2.2. To help the reader, let us first explain the strategy of the argument: we will study the terminal morphism $q\colon \mathcal{L} \to *$, decompose it as a composition

$$q\colon \mathcal{L} \xrightarrow{q^{(0)}} \mathcal{L}^{(0)} \xrightarrow{q^{(1)}} \mathcal{L}^{(1)} \xrightarrow{q^{(2)}} \mathcal{L}^{(2)} \xrightarrow{q^{(3)}} \mathcal{L}^{(3)} \cong * \tag{3.7}$$

and prove that each of the maps $q^{(i)}$ is an equivalence of Kan complexes. The idea is that each ∞-category $\mathcal{L}^{(i)}$ parametrizes diagrams in $\mathcal{T}$ of a certain shape $S^{(i)}$ and with certain data fixed. For $i > 0$, the functors $q^{(i)}\colon \mathcal{L}^{(i-1)} \to \mathcal{L}^{(i)}$ can be interpreted as forgetful maps. The simplicial set $S^{(0)}$ is $\Delta^2 \times \Delta^1$ and corresponds to the shape of diagrams in $\mathcal{E}$ corresponding to triangle in $\mathcal{BO}$ (such as diagram (3.6)). The decreasing sequence of simplicial sets $S^{(0)} \supset S^{(1)} \supset S^{(2)} \supset S^{(3)}$ encode diagrams with fewer and fewer non-fixed data (see definition 3.4.1).

The following picture illustrates the decomposition of the composite functor $\mathcal{L}^{(0)} \to \mathcal{L}^{(3)}$.



(3.8)

In this description, solid arrows stand for morphisms in $\mathcal{T}$ that are fixed within $\mathcal{L}^{(i)}$, whereas dashed arrows indicate morphisms that are allowed to vary in that space. At each step of the composition, the new diagram is obtained from the previous one by removing one 3-simplex in $\mathcal{T}$ (and some simplices of smaller dimension), namely the 3-simplices $\nu\nu^+\sigma^+\tau^+$, $\nu\sigma\sigma^+\tau^+$ and $\nu\sigma\tau\tau^+$, respectively for $q^{(1)}$, $q^{(2)}$ and $q^{(3)}$ (as indicated in grey in the picture). The last ∞-category $\mathcal{L}^{(3)} \cong *$ should be thought of as the fixed data in the $\mathcal{L}^{(i)}$.

The functors $q^{(2)}$ and $q^{(3)}$ are both induced by inner anodyne morphisms and will therefore be trivial Kan fibrations. The case of the functor $q^{(1)}$ is more delicate, and proving that it is also a trivial fibration will constitute the heart of the proof of proposition 3.2.2.

We divide the argument outlined above in three steps: each one amounts to proving that some of the functors $q^{(i)}$ are equivalences. We postpone the most technical parts of the proof to the end of the chapter (section 3.6).

3.3 From slices to functor categories: the functor $q^{(0)}$

The first step is to define the functor $q^{(0)}\colon \mathcal{L} \to \mathcal{L}^{(0)}$ and prove that it is a categorical equivalence. The ∞-category $\mathcal{L}^{(0)}$ will be a slight variation of $\mathcal{L}$, in that these two ∞-categories both parametrize triangles of the form (3.6) with the following data fixed: the morphisms f^+ and g^+ in $\mathcal{BO}$ and the triangle u_0 underlying u of shape (3.5). The two ∞-categories thus share the same objects, the difference being that $\mathcal{L}$ is constructed from the slice ∞-category $\mathcal{BO}_{/f^+}$ whereas $\mathcal{L}^{(0)}$ is obtained from the functor ∞-category $\mathrm{Fun}(\Delta^2, \mathcal{BO})$. More precisely, we define $\mathcal{L}^{(0)}$ as

$$\mathcal{L}^{(0)} = \mathcal{BO}^{\Delta^2} \times_{\mathcal{BO}^{\Lambda^2_2}} \{(f^+, g^+)\} \times_{\mathcal{T}^{\Delta^2}} \{u_0\}. \tag{3.9}$$

Lemma 3.3.1. *There exists an equivalence of ∞-categories $q^{(0)}\colon \mathcal{L} \to \mathcal{L}^{(0)}$. In particular, $\mathcal{L}^{(0)}$ is a Kan complex.*

To construct this equivalence $q^{(0)}$, we first need a comparison between slice ∞-categories and corresponding ∞-categories of diagrams, given by the following lemma.

Lemma 3.3.2. *Let $\mathcal{C}$ be an ∞-category and $p\colon K \to \mathcal{C}$ a diagram. Then there is a canonical equivalence of ∞-categories*

$$\mathcal{C}_{/p} \xrightarrow{\sim} \mathcal{C}^{K^{\triangleleft}} \times_{\mathcal{C}^K} \{p\}.$$

For the sake of completeness, we provide a proof of this folklore result at the end of this chapter, see 3.6.1. We can now proceed to the proof of lemma 3.3.1.

Proof of lemma 3.3.1. First, note that we can write

$$\mathcal{L}^{(0)} = \left(\mathcal{BO}^{\Delta^2} \times_{\mathcal{BO}^{\Delta^{12}}} \{f^+\}\right) \times_{\mathcal{P}} \{u\},$$

where $\mathcal{P}$ denotes the pullback

$$\mathcal{P} = \left(\mathcal{BO}^{\Delta^{02}} \times_{\mathcal{BO}^{\Delta^{\{2\}}}} \{\tau^+\}\right) \times_{\left(\mathcal{T}^{\Delta^{02}} \times_{\mathcal{T}^{\Delta^{\{2\}}}} \{\tau\}\right)} \left(\mathcal{T}^{\Delta^2} \times_{\mathcal{T}^{\Delta^{12}}} \{f\}\right).$$

We define the functor $q^{(0)}\colon \mathcal{L} \to \mathcal{L}^{(0)}$ as the one induced from the commutative square

$$\begin{tikzcd}
\mathcal{BO}_{/f^+} \arrow[r, "\psi'"] \arrow[d, "\xi'"'] & \mathcal{BO}^{\Delta^2} \times_{\mathcal{BO}^{\Delta^{12}}} \{f^+\} \arrow[d, "\xi"] \\
\mathcal{BO}_{/\tau^+} \times_{\mathcal{T}_{/\tau}} \mathcal{T}_{/f} \arrow[r, "\psi"] & \mathcal{P}
\end{tikzcd} \tag{3.10}$$

by taking the fiber at $u \in \mathcal{BO}_{/\tau^+} \times_{\mathcal{T}_{/\tau}} \mathcal{T}_{/f}$.

To prove that $q^{(0)}$ is an equivalence of ∞-categories, we will use that Joyal's model structure is *locally right proper* (although it is not right proper). This property holds for any model structure, and means that for any diagram $d = (X_a \to X_b \leftarrow X_c)$ of fibrant objects in which one of the maps is a fibration, the canonical morphism $X_a \times_{X_b} X_c \to X_a \times^{\mathrm{h}}_{X_b} X_c$ from the pullback to the homotopy pullback is an equivalence. In particular, if a morphism of such diagrams $d \to d'$ is a pointwise weak equivalence, then the induced morphism $\lim d \to \lim d'$ is a weak equivalence.

Since each of the simplicial sets in diagram (3.10) is an ∞-category (hence a fibrant object in Joyal's model structure), to show that $q^{(0)}$ is an equivalence of ∞-categories, it suffices to establish that the following two claims :

(1) ψ and ψ' are categorical equivalences,

(2) ξ and ξ' are isofibrations.

To prove the first claim, we make again use of the argument described in the previous paragraphs. Indeed, the morphism ψ is itself induced from the natural transformation of diagrams

$$\begin{tikzcd}
\mathcal{BO}_{/\tau^+} \arrow[r] \arrow[d, "\psi_a"] & \mathcal{T}_{/\tau} \arrow[d, "\psi_b"] & \mathcal{T}_{/f} \arrow[l] \arrow[d, "\psi_c"] \\
\mathcal{BO}^{\Delta^{02}} \times_{\mathcal{BO}^{\Delta\{2\}}} \{\tau^+\} \arrow[r] & \mathcal{T}^{\Delta^{02}} \times_{\mathcal{T}^{\Delta\{2\}}} \{\tau\} & \mathcal{T}^{\Delta^2} \times_{\mathcal{T}^{\Delta^{12}}} \{f\}. \arrow[l, "\chi"]
\end{tikzcd}$$

Lemma 3.3.2 guarantees that each of the vertical morphism are equivalences. We know that $\mathcal{T}_{/f} \to \mathcal{T}_{/\tau}$ is a right fibration (by the dual of Proposition 2.1.2.1 in [Lur09a]), hence an isofibration.

We now prove that the functor $\chi\colon \mathcal{T}^{\Delta^2} \times_{\mathcal{T}^{\Delta^{12}}} \{f\} \to \mathcal{T}^{\Delta^{02}} \times_{\mathcal{T}^{\Delta\{2\}}} \{\tau\}$ is an isofibration. Let v be an object in $\mathcal{T}^{\Delta^2} \times_{\mathcal{T}^{\Delta^{12}}} \{f\}$ and $\chi(v) \xrightarrow{\sim} \overline{w}$ an equivalence in $\mathcal{T}^{\Delta^{02}} \times_{\mathcal{T}^{\Delta\{2\}}} \{\tau\}$. We want to lift this equivalence to one in $\mathcal{T}^{\Delta^2} \times_{\mathcal{T}^{\Delta^{12}}} \{f\}$. The datum of v is that of a triangle in $\mathcal{T}$ of the form

$$\begin{tikzcd}
 & \sigma \arrow[dr, "f"] & \\
\alpha \arrow[ur] \arrow[rr, "\ell"] & & \tau.
\end{tikzcd} \tag{3.11}$$

The datum of the morphism $\chi(v) \to \overline{w}$ is that of a commutative square of the form

$$\begin{tikzcd}
\alpha \arrow[r, "\ell"] \arrow[d] \arrow[dr] & \tau \arrow[d, equal] \\
\alpha' \arrow[r] & \tau'.
\end{tikzcd} \tag{3.12}$$

As a natural transformation is an equivalence if it so pointwise, the fact that $\chi(v) \to \overline{w}$ is an equivalence translates into the statement that $\alpha \to \alpha'$ is an equivalence. We want to extend the previous diagram into one of the form

$$\begin{tikzcd}
& \sigma \arrow[dr, "f"] \arrow[dd, equal] & \\
\alpha \arrow[ur] \arrow[rr, "\ell"] \arrow[ddd] & & \tau \arrow[ddd, equal] \\
& \sigma' \arrow[ddr, "f"] & \\
& & \\
\alpha' \arrow[uur] \arrow[rr] & & \tau'
\end{tikzcd} \tag{3.13}$$

which represents an equivalence in $\mathcal{T}^{\Delta^2} \times_{\mathcal{T}^{\Delta^{12}}} \{f\}$. We do this construction in several steps : first, by gluing diagrams (3.11) and (3.12) and adding degenerate 2-simplices $\sigma\tau\tau'$ and $\sigma\sigma'\tau'$, we obtain the diagram

$$\begin{tikzcd}
& \sigma \arrow[dr, "f"] \arrow[dd, equal] & \\
\alpha \arrow[ur] \arrow[rr, "\ell"] \arrow[ddd] & & \tau \arrow[ddd, equal] \\
& \sigma' \arrow[ddr, "f"] & \\
& & \\
\alpha' \arrow[rr] & & \tau'.
\end{tikzcd}$$

From this point, the construction of diagram (3.13) is obtained using successive horn fillers in $\mathcal{T}$, that is to say a sequence of choices of solutions to lifting problems, each of the form

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d] & \mathcal{T}. \\
\Delta^n \arrow[ur, dashed] &
\end{tikzcd}$$

First, choosing a filler of the horn of shape Λ^3_2 in $\alpha\sigma\tau\tau'$, we construct the 2-simplex $\alpha\sigma\tau'$. Similarly, by filling the horn Λ^2_1 in $\alpha\sigma\sigma'$, we obtain a morphism $\alpha\sigma'$. Using a filler of the horn Λ^2_1 in $\alpha\sigma\sigma'\tau'$, we get a 2-simplex $\alpha\sigma'\tau'$. Finally, using that the morphism $\alpha\alpha'$ is an equivalence, we can fill the horn Λ^2_0 in $\alpha\alpha'\sigma'$, as well as the horn Λ^3_0 in $\alpha\alpha'\sigma'\tau'$. This yields a diagram of the form (3.13) in which $\alpha \to \alpha'$ is an equivalence; hence an equivalence $v \xrightarrow{\sim} w$ lifting the given morphism $\chi(v) \to \overline{w}$ along χ. This concludes the proof of the first claim.

We now come to the second claim. As ξ' is a right fibration, it is in particular an isofibration. It remains to prove that ξ is also an isofibration. Consider an object $x \in \mathcal{BO}_{/\tau^+} \times_{\mathcal{T}_{/\tau}} \mathcal{T}_{/f}$ and an equivalence $\xi(x) \xrightarrow{\sim} \overline{y}$ in $\mathcal{P}$. We want to construct an equivalence $x \to y$ lifting $\xi(x) \xrightarrow{\sim} \overline{y}$. The data of x is that of a triangle in $\mathcal{BO}$ of the form

$$\begin{tikzcd}
& \sigma^+ \arrow[dr, "f^+"] & \\
\alpha^+ \arrow[ur] \arrow[rr, "\ell^+"] & & \tau^+.
\end{tikzcd} \tag{3.14}$$

The data of the morphism $\xi(x) \to \overline{y}$ is that of a diagram of the form (3.13) and a lift

$$\begin{tikzcd} \alpha^+ \arrow[r, "\ell^+"] \arrow[d] \arrow[dr] & \tau^+ \arrow[d, equal] \\ \alpha'^+ \arrow[r] & \tau'^+ \end{tikzcd} \tag{3.15}$$

of its subdiagram (3.12) along π. By [Lur22, Corollary 01H4], since the morphism χ is an isofibration, the maximal ∞-subgroupoid $\mathcal{P}^{\simeq}$ of $\mathcal{P}$ is given by the limit of the following diagram of ∞-categories

$$\left(\mathcal{BO}^{\Delta^{02}} \times_{\mathcal{BO}^{\Delta^{\{2\}}}} \{\tau^+\}\right)^{\simeq} \longrightarrow \left(\mathcal{T}^{\Delta^{02}} \times_{\mathcal{T}^{\Delta^{\{2\}}}} \{\tau\}\right)^{\simeq} \xleftarrow{\ \chi\ } \left(\mathcal{T}^{\Delta^{2}} \times_{\mathcal{T}^{\Delta^{12}}} \{f\}\right)^{\simeq}.$$

The morphism $\xi(x) \to \overline{y}$ being an equivalence therefore translates into the fact that the morphism $\alpha^+ \to \alpha'^+$ from is an equivalence. Our aim is to extend diagrams (3.14) and (3.15) to obtain a lift

$$\begin{tikzcd} & \sigma^+ \arrow[dr, "f^+"] \arrow[dd, equal] & \\ \alpha^+ \arrow[ur] \arrow[rr, "\ell^+"] \arrow[dd] & & \tau^+ \arrow[dd, equal] \\ & \sigma'^+ \arrow[dr, "f^+"] & \\ \alpha'^+ \arrow[ur] \arrow[rr] & & \tau'^+ \end{tikzcd} \tag{3.16}$$

of the given diagram (3.13) along the functor π. The construction of diagram (3.16) is given by solutions to the same sequence of horn filling problems as that of the proof that χ is an isofibration (in claim (1)); the only difference being that in the present case, the horn filling problems have to be considered relative to the inner fibration π, that is as problems of lifting of the form

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d] & \mathcal{BO} \arrow[d, "\pi"] \\ \Delta^n \arrow[r] \arrow[ur, dashed] & \mathcal{T}. \end{tikzcd}$$

This construction provides an equivalence $x \xrightarrow{\sim} y$ as desired, ensuring that ξ is an isofibration. This concludes the proof of lemma 3.3.1. □

3.4 Anodyne extensions: the functors $q^{(2)}$ and $q^{(3)}$

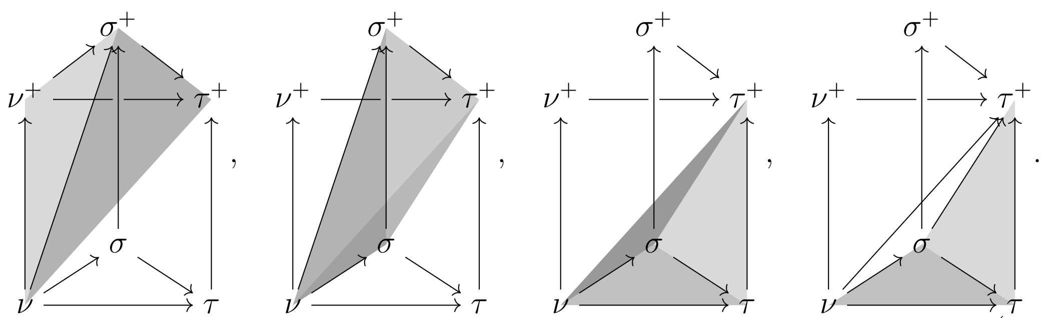
We have defined the functor $q^{(0)} \colon \mathcal{L} \to \mathcal{L}^{(0)}$ and proved it is an equivalence of ∞-groupoids. We now spell out the rest of the decomposition (3.7) of the terminal morphism $q \colon \mathcal{L} \to *$, by defining the ∞-categories $\mathcal{L}^{(i)}$, for $i \in \{1, 2, 3\}$, as well as the functors $q^{(i)}$.

Definition 3.4.1. • First, let $S^{(0)}$ denote the simplicial set $\Delta^2 \times \Delta^1$ with vertices labelled with $\nu, \sigma, \tau, \nu^+, \sigma^+, \nu^+$ as in the diagrams represented in (3.8).

- For $i \in \{1,2,3\}$, we define decreasing subsimplicial sets $S^{(i)}$ of $S^{(0)}$ using the following formulas:

$$\begin{aligned} S^{(1)} &= \Delta^{\nu\nu^+\tau^+} \cup_{\Delta^{\nu\tau^+}} \Delta^{\nu\sigma\tau\tau^+} \cup_{\Delta^{\nu\sigma\tau^+}} \Delta^{\nu\sigma\sigma^+\tau^+}, \\ S^{(2)} &= \Delta^{\nu\nu^+\tau^+} \cup_{\Delta^{\nu\tau^+}} \Delta^{\nu\sigma\tau\tau^+} \cup_{\Delta^{\sigma\tau^+}} \Delta^{\sigma\sigma^+\tau^+}, \\ S^{(3)} &= \Delta^{\nu\nu^+\tau^+} \cup_{\Delta^{\nu\tau^+}} \Lambda^{\nu\sigma\tau\tau^+}_{\tau} \cup_{\Delta^{\sigma\tau^+}} \Delta^{\sigma\sigma^+\tau^+}. \end{aligned} \tag{3.17}$$

For $i \in \{0,1,2\}$, the simplicial set $S^{(i)}$ will encode the shape of the diagrams parametrized by $\mathcal{L}^{(i)}$, whereas $S^{(3)}$ will describe the shape of diagrams that are fixed within $\mathcal{L}^{(i)}$. The simplicial sets $S^{(0)}, \ldots, S^{(3)}$ can be pictured as



(3.18)

We define $\mathcal{L}^{(3)}$ as the terminal ∞-groupoid ; we think of its unique object as the diagram $S^{(3)} \to \mathcal{T}$ given by the data (f^+, g^+, u_0) that we fixed earlier on. The inclusion $j^{(i)} \colon S^{(i)} \subset S^{(i-1)}$ induces a forgetful functor $p^{(i)} \colon \mathcal{T}^{S^{(i-1)}} \to \mathcal{T}^{S^{(i)}}$ that we use to define the ∞-categories

$$\begin{aligned} \mathcal{L}^{(1)} &= \mathcal{T}^{S^{(1)}} \times_{\mathcal{T}^{S^{(3)}}} \{(f^+, g^+, u_0)\}, \\ \mathcal{L}^{(2)} &= \mathcal{T}^{S^{(2)}} \times_{\mathcal{T}^{S^{(3)}}} \{(f^+, g^+, u_0)\}. \end{aligned}$$

Recall that $\mathcal{L}^{(0)}$ was defined by formula (3.9) using terms of diagrams with values in $\mathcal{BO}$. Nevertheless, the following lemma ensures that $\mathcal{L}^{(0)}$ actually admits a simple equivalent description in terms of diagrams in $\mathcal{T}$, following the above pattern for $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$.

Lemma 3.4.2. *We have a canonical equivalence of ∞-groupoids*

$$\mathcal{L}^{(0)} \simeq \mathcal{T}^{S^{(0)}} \times_{\mathcal{T}^{S^{(3)}}} \{(f^+, g^+, u_0)\}.$$

Proof. It follows from its definition that the ∞-groupoid $\mathcal{L}^{(0)}$ fits into the commutative diagram

$$\begin{tikzcd}
\mathcal{L}^{(0)} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & * \arrow[d] & \\
\mathcal{BO}^{\Delta^2} \arrow[r] \arrow[d] & \mathcal{BO}^{\Lambda^2_2} \times_{\mathcal{T}} \mathcal{T}^{\Delta^2} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{BO}^{\Lambda^2_2} \arrow[d] \\
\mathcal{T}^{\Delta^2 \times \Delta^1} \arrow[r] & \mathcal{T}^{S(3)} \arrow[r] & \mathcal{T}^{\Lambda^2_2 \times \Delta^1}
\end{tikzcd}$$

in which the upper left and the bottom right squares are cartesian. To prove the lemma, it therefore suffices to show that the bottom outer square is cartesian. Note that both vertical maps $\mathcal{BO}^{\Lambda^2_2} \to \mathcal{T}^{\Lambda^2_2 \times \Delta^1}$ and $\mathcal{BO}^{\Delta^2} \to \mathcal{T}^{\Delta^2 \times \Delta^1}$ are subcategories, meaning they are monomorphisms that are inner fibrations. This implies that the morphism $\mathcal{BO}^{\Lambda^2_2} \times_{\mathcal{T}^{\Lambda^2_2 \times \Delta^1}} \mathcal{T}^{\Delta^2} \to \mathcal{T}^{\Delta^2 \times \Delta^1}$ is also the inclusion of a subcategory, hence it is enough to verify that the two subcategories $\mathcal{BO}^{\Delta^2}$ and $\mathcal{BO}^{\Lambda^2_2} \times_{\mathcal{T}^{\Lambda^2_2 \times \Delta^1}} \mathcal{T}^{\Delta^2}$ have the same objects and morphisms. An object in $\mathcal{BO}^{\Delta^2}$ (respectively in $\mathcal{BO}^{\Lambda^2_2} \times_{\mathcal{T}^{\Lambda^2_2 \times \Delta^1}} \mathcal{T}^{\Delta^2}$) is a diagram

$$\begin{tikzcd}
 & \alpha_1^+ \arrow[dr, "t"] & \\
\alpha_0^+ \arrow[ur, "s"] \arrow[rr, "w"] & & \alpha_2^+ \\
 & \alpha_1 \arrow[uu] \arrow[dr] & \\
\alpha_0 \arrow[uu] \arrow[ur] \arrow[rr] & & \alpha_2 \arrow[uu]
\end{tikzcd}$$

in $\mathcal{T}$ in which the maps $\alpha_0\alpha_0^+$, $\alpha_1\alpha_1^+$ and $\alpha_2\alpha_2^+$ are objects in $\mathcal{BO}$ and such that the morphisms s, t and w (respectively only t and w) lie p_1-above $\mathrm{id}_{\langle 1 \rangle}$ and are compatible with extension (conditions of definition 2.4.1). The key observation is that whenever t and w both satisfy these properties, then so does s; this fact implies that the two subcategories have the same objects. One can use a similar argument to show that the same is true for morphisms of these two subcategories, as desired.

□

Therefore, the functors $q^{(i)}$ and the ∞-categories $\mathcal{L}^{(i)}$ fit into the commutative diagram

$$\begin{tikzcd}
\mathcal{L}^{(0)} \arrow[r, "q^{(1)}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{L}^{(1)} \arrow[r, "q^{(2)}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{L}^{(2)} \arrow[r, "q^{(3)}"] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{L}^{(3)} \cong * \arrow[d, "{(f^+, g^+, u_0)}"] \\
\mathcal{T}^{S^{(0)}} \arrow[r, "p^{(1)}"] & \mathcal{T}^{S^{(1)}} \arrow[r, "p^{(2)}"] & \mathcal{T}^{S^{(2)}} \arrow[r, "p^{(3)}"] & \mathcal{T}^{S^{(3)}},
\end{tikzcd} \tag{3.19}$$

where all squares are cartesian.

We now claim that the inclusions $j^{(2)}\colon S^{(1)} \subset S^{(2)}$ and $j^{(3)}\colon S^{(2)} \subset S^{(3)}$ are inner anodyne. For the latter morphism, this is obvious from the formulas (3.17), as $j^{(3)}$ is obtained as a pushout of the inner anodyne map $\Lambda^{\nu\sigma\tau\tau^+}_{\tau} \subset \Delta^{\nu\sigma\tau\tau^+}$. For the former map, note that we can write $j^{(2)}$ as a pushout of the composition

$$\Delta^{\nu\sigma\tau^+} \cup_{\Delta^{\sigma\tau^+}} \Delta^{\sigma\sigma^+\tau^+} \ \subset \ \Lambda^{\nu\sigma\sigma^+\tau^+}_{\sigma} \ \subset \ \Delta^{\nu\sigma\sigma^+\tau^+},$$

in which both maps are inner anodyne. This shows the claim. We therefore obtain that the induced functors $p^{(2)}$ and $p^{(3)}$ are trivial Kan fibrations. As every square in diagram (3.19) is cartesian, we deduce that the functors $q^{(2)}$ and $q^{(3)}$ are also trivial Kan fibrations. In particular, we see that $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ are contractible Kan complexes.

3.5 Existence and uniqueness of factorizations: the functor $q^{(1)}$

In order to prove theorem 3.2.2, it only remains to prove that $q^{(1)}$ is a trivial Kan fibration, which is the main step of the proof. Since the inclusion $j^{(1)}$ restricts to a bijection $S^{(1)}_0 \subset S^{(2)}_0$ on the sets of 0-simplices, by [Rez22, Proposition 40.6 and footnote 30], we obtain that $p^{(1)}\colon \mathcal{L}^{(1)} \to \mathcal{L}^{(2)}$ is an isofibration. Since we already know that both $\mathcal{L}^{(1)}$ and $\mathcal{L}^{(2)}$ are Kan complexes, it follows that $p^{(1)}$ is a Kan fibration. Since the squares in diagram (3.19) are cartesian, we deduce that $q^{(1)}$ is also a Kan fibration between Kan complexes. To see that it is a weak equivalence, we have to show that all of its fibers are contractible, which is asserted in the following proposition.

Proposition 3.5.1. *Let $d\colon S^{(1)} \to \mathfrak{T}$ be a diagram in $\mathcal{L}^{(1)}$. Then the fiber $\mathcal{L}^{(0)}_d$ of $q^{(1)}$ at d is contractible.*

Proof. First, we claim that the question can be restricted to the full subdiagram of d on the objects ν, ν^+, σ^+ and τ^+. Indeed, if we let $S^{(1)'}$ denote the subsimplicial set $\Delta^{\nu\nu^+\tau^+} \cup_{\Delta^{\nu\tau^+}} \Delta^{\nu\sigma^+\tau^+}$ of $S^{(0)'} := \Delta^{\nu\nu^+\sigma^+\tau^+}$ encoding commutative squares of the form

$$\begin{tikzcd} \nu^+ \arrow[r] & \tau^+ \\ \nu \arrow[u] \arrow[r] \arrow[ur] & \sigma^+ \arrow[u] \end{tikzcd} \tag{3.20}$$

we observe that $j^{(1)}\colon S^{(1)} \subset S^{(0)}$ can be written as the pushout

$$\begin{tikzcd} S^{(1)'} \arrow[r] \arrow[d] & S^{(0)'} \arrow[d] \\ S^{(1)} \arrow[r, "j^{(1)}"'] \arrow[ur, phantom, "\ulcorner"] & S^{(0)}, \end{tikzcd} \tag{3.21}$$

of the inclusion. Therefore we can identify the fiber of $q^{(1)}$ at d as

$$\mathcal{L}_d^{(0)} \simeq \mathfrak{T}^{S(0)'} \times_{\mathfrak{T}^{S(1)'}} \{d'\},$$

where $d' = d|_{S(1)'}$, thus proving the claim.

Note that the diagram d', which is of shape (3.20), is essentially determined by the fixed data (f^+, g^+, u_0). In particular, in this diagram, the morphism $\sigma^+\tau^+$ is the edge f^+ constructed in section 3.1 and the 2-simplices $\nu\sigma^+\tau^+$ and $\nu\nu^+\tau^+$ are given. The proof of the proposition thus consists in showing that, from this data, the remaining simplices $\nu^+\sigma^+$, $\nu\nu^+\sigma^+$, $\nu^+\sigma^+\tau^+$ and $\nu\nu^+\sigma^+\tau^+$ can be constructed in an essentially unique way.

Until now, the simplices were written in the ∞-category $\mathfrak{T}$; we need to reformulate the problem in terms of diagrams with values in $\mathcal{O}^{\otimes}$. The diagram d' of shape (3.20) corresponds to a certain diagram $d'_{\mathcal{O}} \colon K \to \mathcal{O}^{\otimes}$, which can be pictured as

$$\begin{tikzcd}
& V_0^+ \arrow[rr] \arrow[dd] & & T_0^+ \arrow[dd] & & & 1 \arrow[rr] \arrow[dd] & & 3 \arrow[dd] \\
V_0 \arrow[ur, hook] \arrow[rr] \arrow[urrr] \arrow[dd] & & S_0^+ \arrow[ur] \arrow[dd] & & \overset{\text{def}}{=} & 0 \arrow[ur, hook] \arrow[rr] \arrow[urrr] \arrow[dd] & & 2 \arrow[ur] \arrow[dd] & \\
& V_1^+ \arrow[dl] & & T_1^+ \arrow[ll] \arrow[dl] \arrow[dlll] & & & 6 \arrow[dl] & & 4. \arrow[ll] \arrow[dl] \arrow[dlll] \\
V_1 & & S_1^+ \arrow[ll] & & & 7 & & 5 \arrow[ll] &
\end{tikzcd} \tag{3.22}$$

Here, in order to sometimes simplify notations, we denote the objects V_0, V_0^+, S_0^+, T_0^+, T_1^+, S_1^+, V_1^+, V_1 as the integers $0, \dots, 7$, in the same order. This way, we can write the simplicial set indexing $d'_{\mathcal{O}}$ as the subsimplicial set

$$K = \Delta^{013467} \cup_{\Delta^{0347}} \Delta^{023457}$$

of Δ^7. The fiber $\mathcal{L}_d^{(0)}$ is therefore canonically equivalent to the space $\mathcal{L}_d^{(0)} \simeq (\mathcal{O}^{\otimes})^{\Delta^7} \times_{(\mathcal{O}^{\otimes})^K} \{d'_{\mathcal{O}}\}$ parametrizing extensions of the diagram $d'_{\mathcal{O}}$ to a 7-simplex.

The key step of the proof concerns the space of lifts of the upper part of diagram (3.22), namely the subdiagram indexed by the full subsimplicial set $K_0 \subseteq K$ on the objects V_0, V_0^+, S_0^+ and T_0^+. Note that this simplicial set is isomorphic to $\Delta^1 \times \Delta^1$. Define $\mathcal{Z}$ as the fiber $(\mathcal{O}^{\otimes})^{\Delta^3} \times_{(\mathcal{O}^{\otimes})^{K_0}} \{d'_{\mathcal{O}}|_{K_0}\}$; this ∞-category parametrizes extensions of the diagram $d'_{\mathcal{O}}|_{K_0} \colon K_0 \to \mathcal{O}$ to a 3-simplex $V_0V_0^+S_0^+T_0^+$. Since the inclusion $K_0 \subset \Delta^3$ is a bijection on objects, the induced functor $(\mathcal{O}^{\otimes})^{\Delta^3} \to (\mathcal{O}^{\otimes})^{K_0}$ is an isofibration and therefore the ∞-category $\mathcal{Z}$ is a space. We will show the following intermediate result.

Claim 3.5.2. *The space $\mathcal{Z} = (\mathcal{O}^{\otimes})^{\Delta^3} \times_{(\mathcal{O}^{\otimes})^{K_0}} \{d'_{\mathcal{O}}|_{K_0}\}$ of lifts of the upper part of diagram (3.22) is contractible.*

The argument relies on the observation that the diagram $K_0 \to \mathcal{O}^{\otimes} \to \mathbb{F}_*$ has a decomposition $p \circ d'_{\mathcal{O}}|_{K_0} = d^-_{\mathbb{F}_*} \oplus d^+_{\mathbb{F}_*}$ given by

$$\begin{tikzcd} p_0(V_0)^+ \arrow[r] & p_0(T_0^+) \\ p_0(V_0) \arrow[r] \arrow[u, hook] & p_0(S_0^+) \arrow[u] \end{tikzcd} = \begin{tikzcd} \langle n \rangle \arrow[r] & \langle k \rangle \\ \langle n \rangle \arrow[r] \arrow[u, equal] & \langle m \rangle \arrow[u] \end{tikzcd} \oplus \begin{tikzcd} \langle 1 \rangle \arrow[r, equal] & \langle 1 \rangle \\ \langle 0 \rangle \arrow[r] \arrow[u, hook] & \langle 1 \rangle \arrow[u, equal] \end{tikzcd}, \tag{3.23}$$

where $\oplus$ stands for the operation of pointwise disjoint union of diagrams in $\mathbb{F}_*$. Using the identity maps in (3.23), one readily sees that both diagrams $d^-_{\mathbb{F}_*}$ and $d^+_{\mathbb{F}_*}$ extend uniquely to 3-simplices $\widetilde{d}^-_{\mathbb{F}_*}$ and $\widetilde{d}^+_{\mathbb{F}_*}$ in $\mathbb{F}_*$, which implies that the diagram $p \circ d'_{\mathcal{O}}|_{K_0}$ also extends uniquely to a 3-simplex, namely $\widetilde{d}'_{\mathbb{F}_*} := \widetilde{d}^-_{\mathbb{F}_*} \oplus \widetilde{d}^+_{\mathbb{F}_*}$. In particular, any diagram $\Delta^3 \to \mathcal{O}^{\otimes}$ in $\mathcal{Z}$ will be a lift of $\widetilde{d}'_{\mathbb{F}_*}$. This shows that we can rewrite the space $\mathcal{Z}$ as

$$\mathcal{Z} \simeq (\mathcal{O}^{\otimes})^{\Delta^3} \times_{(\mathcal{O}^{\otimes})^{K_0}} \{d'_{\mathcal{O}}|_{K_0}\} \times_{\mathbb{F}_*^{\Delta^3}} \{\widetilde{d}'_{\mathbb{F}_*}\}.$$

We will make use of decomposition (3.23) to obtain a splitting of the space $\mathcal{Z}$, using the next lemma. Recall that a simplicial set is said to be *braced* if every face of a nondegenerate simplex remains nondegenerate [Lur22, Tag 00XU].

Lemma 3.5.3. *Let $\mathcal{O}^{\otimes}$ be any ∞-operad. Let J be a braced simplicial set and $I \subseteq J$ be a subsimplicial set. Consider two diagrams q and r making the following square commute:*

$$\begin{tikzcd} I \arrow[r, "r"] \arrow[d, hook] & \mathcal{O}^{\otimes} \arrow[d, "p"] \\ J \arrow[r, "q"] & \mathbb{F}_* \end{tikzcd} \tag{3.24}$$

and assume that q decomposes as a disjoint union $q = \oplus_{i=1}^n q_i$ of diagrams $J \to \mathbb{F}_*$. Then there exists a decomposition $r \simeq \oplus_{i=1}^n r_i$ such that the ∞-category of lifts in the square (3.24) splits as the following direct product:

$$(\mathcal{O}^{\otimes})^J \times_{(\mathcal{O}^{\otimes})^I} \{r\} \times_{\mathbb{F}_*^J} \{q\} \simeq \prod_{i=1}^n (\mathcal{O}^{\otimes})^J \times_{(\mathcal{O}^{\otimes})^I} \{r_i\} \times_{\mathbb{F}_*^J} \{q_i\}. \tag{3.25}$$

The proof of this lemma is given at the end of this chapter, in section 3.6.2.

We can now complete the proof of lemma 3.5.3. Through the first of the equivalences in (3.33), the object $r \in (\mathcal{O}^{\otimes})^I$ is identified with an object $\oplus_i r_i$. Now observe that $\mathcal{D}$ fits in a diagram of pullback squares

$$\begin{tikzcd} \mathcal{D} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^{\otimes})^J \times_{\mathbb{F}_*^J} \{q\} \arrow[r] \arrow[d, "\rho'"] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^{\otimes})^J \arrow[d, "\rho"] \\ * \arrow[r, "r"] \arrow[dr, equal, bend right] & (\mathcal{O}^{\otimes})^I \times_{\mathbb{F}_*^I} \{q|_I\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^{\otimes})^I \times_{\mathbb{F}_*^I} \mathbb{F}_*^J \arrow[d] \\ & * \arrow[r, "q"] & \mathbb{F}_*^J. \end{tikzcd}$$

Since the vertical functor ρ is an isofibration, so is its pullback ρ'. Therefore the top left pullback square is invariant under equivalence of ∞-categories. Using the above two decompositions, we deduce that $\mathcal{D}$ itself can be written as a product

$$\mathcal{D} \simeq \prod_{i=1}^{n} (\mathcal{O}^{\otimes})^{J} \times_{\mathbb{F}_*^J} \{q_i\} \underset{(\mathcal{O}^{\otimes})^I \times_{\mathbb{F}_*^I} \{q_i|_I\}}{\times} \{r_i\} \simeq \prod_{i=1}^{n} (\mathcal{O}^{\otimes})^{J} \times_{(\mathcal{O}^{\otimes})^I} \{r_i\} \times_{\mathbb{F}_*^J} \{q_i\},$$

as desired. □

We now come back to the proof of proposition 3.5.1. Using the previous lemma, since Δ^3 is braced, we obtain a decomposition $d'_{\mathcal{O}}|_{K_0} = d^-_{\mathcal{O}} \oplus d^+_{\mathcal{O}}$ lifting that of equation (3.23) and a corresponding splitting $\mathcal{Z} = \mathcal{Z}^- \times \mathcal{Z}^+$, with components given by

$$\mathcal{Z}^{\pm} = (\mathcal{O}^{\otimes})^{\Delta^3} \times_{(\mathcal{O}^{\otimes})^{K_0}} \left\{ d^{\pm}_{\mathcal{O}} \right\} \times_{\mathbb{F}_*^{\Delta^3}} \left\{ \widetilde{d}^{\pm}_{\mathbb{F}_*} \right\}.$$

As before, we note that any diagram $\Delta^3 \to \mathcal{O}^{\otimes}$ lifting $d^{\pm}_{\mathcal{O}}$ will automatically be a lift of $\widetilde{d}^{\pm}_{\mathbb{F}_*}$. Therefore, the space $\mathcal{Z}^{\pm}$ is equivalent to $(\mathcal{O}^{\otimes})^{\Delta^3} \times_{(\mathcal{O}^{\otimes})^{K_0}} \{d^{\pm}_{\mathcal{O}}\}$. We note that the arrow in $d^-_{\mathcal{O}}$ that lifts the left vertical map $\mathrm{id}_{\langle n \rangle}$ in diagram (3.23) is an equivalence, since by assumption $V_0 \to V_0^+$ is semi-inert. Similarly, the arrow in $d^+_{\mathcal{O}}$ that lifts the right vertical map $\mathrm{id}_{\langle 1 \rangle}$ in diagram (3.23) is necessarily an equivalence, by construction of the map $S_0^+ \to T_0^+$. We claim that these properties force the spaces $\mathcal{Z}^-$ and $\mathcal{Z}^+$ to be contractible. To see this, we need the following version of Joyal's lifting theorem.

Lemma 3.5.4. *Let $\mathcal{C}$ be an ∞-category and α an equivalence in $\mathcal{C}$. Consider an outer horn $\overline{\alpha}\colon \Lambda_0^n \to \mathcal{C}$, with $n \geqslant 2$, whose restriction along $\Delta^1 \xrightarrow{\langle 01 \rangle} \Lambda_0^n$ is α. Then the fiber $\mathcal{C}^{\Delta^n} \times_{\mathcal{C}^{\Lambda_0^n}} \{\overline{\alpha}\}$ parametrizing extensions of the form*

$$\begin{tikzcd} \Delta^1 \arrow[r, hook, "\langle 01 \rangle"'] \arrow[rr, bend left, "\alpha"] & \Lambda_0^n \arrow[r, "\overline{\alpha}"] \arrow[d, hook] & \mathcal{C} \\ & \Delta^n \arrow[ur, dashed] & \end{tikzcd} \tag{3.26}$$

is a contractible ∞-groupoid.

Proof of lemma 3.5.4. We want to show that any morphism $\beta\colon \partial\Delta^m \to \mathcal{C}^{\Delta^n} \times_{\mathcal{C}^{\Lambda_0^n}} \{\overline{\alpha}\}$ extends to an m-simplex, for all $m \geqslant 0$. This problem is equivalent to finding a lift in the commutative square

$$\begin{tikzcd} \Lambda_0^n \arrow[r, "\mathrm{diag} \circ \overline{\alpha}"] \arrow[d, hook] & \mathcal{C}^{\Delta^m} \arrow[d, "i^*"] \\ \Delta^n \arrow[r, "\beta'"] \arrow[ur, dashed] & \mathcal{C}^{\partial\Delta^m} \end{tikzcd}$$

where diag is the diagonal functor $\mathcal{C} \to \mathcal{C}^{\Delta^m}$, i^* is the inner fibration induced by the inclusion $i\colon \partial\Delta^m \to \Delta^m$ and β' is adjoint to $\partial\Delta^m \xrightarrow{\beta} \mathcal{C}^{\Delta^n} \times_{\mathcal{C}^{\Lambda_0^n}} \{\overline{\alpha}\} \to \mathcal{C}^{\Delta^n}$. As α is an equivalence, so is $\mathrm{diag} \circ \alpha\colon \Delta^1 \to \mathcal{C}^{\Delta^m}$. Therefore, using Joyal's lifting theorem [Lur22, Theorem 019F], we obtain the existence of lifts in the above square, as desired. □

From the above argument and the previous lemma, we deduce that the space $\mathcal{Z}$ is contractible. This proves claim 3.5.2. To finish the proof of proposition 3.5.1, consider the pushout $\widetilde{K} = K \cup_{K_0} \Delta^3$ and the inclusion $\iota\colon \widetilde{K} \to \Delta^7$. The fibers $\mathcal{L}_d^{(0)}$ and $\mathcal{Z}$ fit in the commutative diagram

$$\begin{tikzcd}
\mathcal{L}_d^{(0)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{Z} \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \{d'_{\mathcal{O}}\} \arrow[d] \\
(\mathcal{O}^{\otimes})^{\Delta^7} \arrow[r, "\iota^*"] & (\mathcal{O}^{\otimes})^{\widetilde{K}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{O}^{\otimes})^{K} \arrow[d] \\
 & (\mathcal{O}^{\otimes})^{\Delta^3} \arrow[r] & (\mathcal{O}^{\otimes})^{K_0}.
\end{tikzcd}$$

Here, both the right outer and the bottom right squares are cartesian, therefore so is the top right square. As the top outer square is cartesian, so must be the top right square. To complete the proof that $\mathcal{L}_d^{(0)}$ is contractible, we need a careful analysis of the morphism ι. The result makes use of the notion of *right anodyne morphism* recalled in the appendix in definition A.1.5 and writes as follows.

Lemma 3.5.5. *Consider the simplicial set* Δ^7 *as a marked simplicial set, with* $6 \to 7$ *as the only nondegenerate marked edge. Then the inclusion* $\iota\colon \widetilde{K} \longrightarrow \Delta^7$ *is right marked anodyne.*

This result is proved using a tedious explicit calculation that we defer to the end of this chapter, in section 3.6.3.

Using the previous lemma and the observation that the morphism 67 in diagram $d'|_{\mathcal{O}}$ is an equivalence, it then follows from lemma 3.5.4 that the functor $\mathcal{L}_d^{(0)} \to \mathcal{Z}$ induced by ι^* is an equivalence.

This concludes the proof of proposition 3.5.1, hence that of theorem 3.2.2.

3.6 Proof of technical lemmas

In this section, we complete the proof of theorem 2.5.1 by providing proofs to lemmas 3.3.2, 3.5.3 and 3.5.5.

3.6.1 Proof of lemma 3.3.2

Proof of lemma 3.3.2. Let W be a simplicial set. By definition of the slice ∞-category, we have a natural bijection

$$\mathrm{Hom}(W, \mathcal{C}_{/p}) = \mathrm{Hom}_{K_/}(W \star K, \mathcal{C}),$$

where the index $K_/$ denotes the subset of those morphisms $W \star K \to \mathcal{C}$ that restrict to p on K. On the other hand, we have a natural bijection

$$\mathrm{Hom}(W, \mathcal{C}^{K^{\triangleleft}} \times_{\mathcal{C}^K} \{p\}) = \mathrm{Hom}_{K_/}(\overline{W}, \mathcal{C}),$$

where we use the simplicial set

$$\overline{W} = (W \times K^{\triangleleft}) \coprod_{W \times K} K.$$

We now construct a categorical equivalence $\varphi_W : \overline{W} \to W \star K$, natural in W. It is obtained by the universal property of the pushout $\overline{W}$, induced by the canonical inclusion $K \to W \star K$ and a certain morphism $W \times K^{\triangleleft} \to W \star K$. To describe the latter, recall that maps from a simplicial set X to $W \star K$ can be identified with triples of morphisms $(X \to \Delta^1, X_0 \to W, X_1 \to K)$, where $X_i = \{i\} \times_{\Delta^1} X$ is the fiber at i. Using this description, the morphism $W \times K^{\triangleleft} \to W \star K$ corresponds to the triple

$$(\mathrm{can} \circ \mathrm{proj} \colon W \times K^{\triangleleft} \to K^{\triangleleft} \to \Delta^1, \quad W \times \Delta^0 \cong W, \quad \mathrm{proj} \colon W \times K \to K).$$

We now prove that φ_W is a categorical equivalence. The argument relies on the fact that the canonical morphism $c_{A,B} \colon A \diamond B \to A \star B$, comparing the two join constructions, is a categorical equivalence for all simplicial sets A, B [Lur22, Theorem 01HV]. Observe that $\overline{W}$ fits in a commutative diagram

$$\begin{tikzcd}
W \times K \arrow[r, hook] \arrow[d, "\mathrm{proj}"] & W \times (\Delta^0 \diamond K) \arrow[r, "\mathrm{id}_W \times c_{\Delta^0,K}"'] \arrow[d] & W \times K^{\triangleleft} \arrow[d] \\
K \arrow[r] \arrow[ur, phantom, "\ulcorner"] & W \diamond K \arrow[r, "\overline{c}_{W,K}"] \arrow[ur, phantom, "\ulcorner"] & \overline{W}
\end{tikzcd}$$

in which all three squares are cocartesian. Since the top horizontal maps $W \times K \to W \times (\Delta^0 \diamond K)$ and $W \times K \to W \times K^{\triangleleft}$ are monomorphisms, the left and outer squares are categorical pushout squares; therefore, so must be the right one. As the top right morphism $\mathrm{id}_W \times c_{\Delta^0,K}$ is a categorical equivalence, so is $\overline{c}_{W,K}$. Now observing that the comparison equivalence $c_{W,K} \colon W \diamond K \to W \star K$ factors as

$$W \diamond K \xrightarrow{\overline{c}_{W,K}} \overline{W} \xrightarrow{\varphi_W} W \star K,$$

we conclude that φ_W is a categorical equivalence.

The morphisms φ_W for varying W induce a functor of ∞-categories $\varphi \colon \mathcal{C}_{/p} \to \mathcal{C}^{K^{\triangleleft}} \times_{\mathcal{C}^K} \{p\}$. To prove that φ is an equivalence, we will show that for each simplicial set W, the induced morphism

$$\varphi_* \colon \pi_0 \left(\mathrm{Fun}(W, \mathcal{C}_{/p})^{\simeq} \right) \longrightarrow \pi_0 \left(\mathrm{Fun}(W, \mathcal{C}^{K^{\triangleleft}} \times_{\mathcal{C}^K} \{p\})^{\simeq} \right) \tag{3.27}$$

is a bijection. By [Lur22, Tag 01KV], we have a canonical bijection

$$\pi_0 \left(\mathrm{Fun}(W, \mathcal{C}_{/p})^{\simeq} \right) \cong \pi_0 \left(\mathrm{Fun}_{K/}(W \star K, \mathcal{C})^{\simeq} \right).$$

We claim that one has a similar bijection for the target of φ_*, namely:

Claim. There is a canonical bijection

$$\pi_0 \left(\mathrm{Fun} \left(W, \mathcal{C}^{K^{\triangleleft}} \times_{\mathcal{C}^K} \{p\} \right)^{\simeq} \right) \cong \pi_0 \left(\mathrm{Fun}_{K/}(\overline{W}, \mathcal{C})^{\simeq} \right). \tag{3.28}$$

Assuming this claim, we see that the morphism φ_* from (3.27) corresponds, under the previous two bijections, to the map

$$\varphi_W^*\colon \pi_0\left(\mathrm{Fun}_{K_/}(W\star K,\mathcal{C})^{\simeq}\right)\longrightarrow\pi_0\left(\mathrm{Fun}_{K_/}(\overline{W},\mathcal{C})^{\simeq}\right),$$

induced by φ_W. By assumption, φ_W is a categorical equivalence compatible with restricting to K; thus φ_W^* is a bijection, for all W, as desired.

To complete the proof, we now prove claim (3.28). Let α_0,α_1 be two functors $W\to\mathcal{C}^{K^{\triangleleft}}\times_{\mathcal{C}^K}\{p\}$ and let $\overline{\alpha}_0,\overline{\alpha}_1$ denote the corresponding objects in $\mathrm{Fun}_{K_/}(\overline{W},\mathcal{C})$ under the bijection $\mathrm{Hom}\left(W,\mathcal{C}^{K^{\triangleleft}}\times_{\mathcal{C}^K}\{p\}\right)\cong\mathrm{Hom}_{K_/}(\overline{W},\mathcal{C})$. We wish to prove that α_0 and α_1 are equivalent if and only if $\overline{\alpha}_0$ and $\overline{\alpha}_1$ are. To this end, we will use a characterization of equivalences in functor categories. Consider a categorical mapping cylinder of W, that is a factorization of $(\mathrm{id}_W,\mathrm{id}_W)$ of the form

$$W\amalg W\xhookrightarrow{(s_0,s_1)}RW\xrightarrow[\sim]{p}W \tag{3.29}$$

where p is a categorical equivalence and (s_0,s_1) is a monomorphism. From [Lur22, Corollary 01KD], we know that the objects α_0 and α_1 are equivalent if and only if the following condition is satisfied:

(1) there exists $\alpha\colon RW\to\mathcal{C}^{K^{\triangleleft}}\times_{\mathcal{C}^K}\{p\}$ such that $\alpha\circ s_0=f_0$ and $\alpha\circ s_1=f_1$.

By definition of the functor $X\mapsto\overline{X}$, one observes that the latter condition is equivalent to the following:

(2) there exists some $\alpha'\colon\overline{RW}\to\mathcal{C}$ satisfying $\alpha'\circ\overline{s_0}=\overline{f}_0$ and $\alpha'\circ\overline{s_1}=\overline{f}_1$.

Using again [Lur22, Corollary 01KD], one sees that this last condition is verified if and only if the objects $\overline{f}_0$ and $\overline{f}_1$ are equivalent, provided that the factorization

$$\overline{W}\amalg_K\overline{W}\xrightarrow{(\bar{s}_0,\bar{s}_1)}\overline{RW}\xrightarrow{\bar{p}}\overline{W} \tag{3.30}$$

is a categorical mapping cylinder for $\overline{W}$ relative to K; the proof of this last fact is an easy verification. This proves claim (3.28) and finishes the proof of lemma 3.3.2.

∎

3.6.2 Proof of lemma 3.5.3

Proof of lemma 3.5.3. For ease of notation, let $\mathcal{D}$ denote the left hand side of (3.25). Consider a diagram $X\in\mathcal{D}$. For each vertex $j\in J$, we can write the object $X(j)$ in the form $\oplus_i X(j)_i$, with the resulting diagram X_i lying over q_i. Then, for each 1-simplex $f\colon j_0\to j_1$ of J, using the definition of ∞-operads, we see that the space $\mathrm{Map}_{\mathcal{O}}^{q(f)}(X(j_0),X(j_1))$ decomposes canonically as $\prod_i\mathrm{Map}_{\mathcal{O}}^{q_i(f)}(X(j_0)_i,X(j_1)_i)$. Therefore, up to equivalence, we can write the morphism $X(f)$ as a disjoint union $\oplus_i X(f)_i$ with each component lying over $q_i(f)$.

We now consider the general case. Let σ be a simplex of J of dimension n and let $\alpha\colon \mathrm{Sp}^n \hookrightarrow \Delta^n$ denote the spine inclusion. Since Sp^n is 1-skeletal, the previous part of the proof gives a decomposition of the space parametrizing diagrams $\mathrm{Sp}^n \to \mathcal{O}^\otimes$ lifting $q(\sigma \circ \alpha)$ as a product

$$\prod_{i=1}^{n} (\mathcal{O}^\otimes)^{\mathrm{Sp}^n} \times_{\mathbb{F}_*^{\mathrm{Sp}^n}} \{q_i(\sigma \circ \alpha)\} \xrightarrow{\simeq} (\mathcal{O}^\otimes)^{\mathrm{Sp}^n} \times_{\mathbb{F}_*^{\mathrm{Sp}^n}} \{q(\sigma \circ \alpha)\}, \tag{3.31}$$

with the equivalence given by disjoint union. Now, as α is anodyne, we have a canonical equivalence between the spaces of diagrams $(\mathcal{O}^\otimes)^{\Delta^n} \xrightarrow{\sim} (\mathcal{O}^\otimes)^{\mathrm{Sp}^n}$, from which we can extend (3.31) to an equivalence

$$\prod_{i=1}^{n} (\mathcal{O}^\otimes)^{\Delta^n} \times_{\mathbb{F}_*^{\Delta^n}} \{q_i(\sigma)\} \xrightarrow{\simeq} (\mathcal{O}^\otimes)^{\Delta^n} \times_{\mathbb{F}_*^{\Delta^n}} \{q(\sigma)\}. \tag{3.32}$$

Claim 3.6.1. *Using the previous equivalence for every simplex σ of I and J, we obtain two decompositions*

$$\prod_{i=1}^{n} (\mathcal{O}^\otimes)^{I} \times_{\mathbb{F}_*^{I}} \{q_i|_I\} \simeq (\mathcal{O}^\otimes)^{I} \times_{\mathbb{F}_*^{I}} \{q|_I\}, \qquad \prod_{i=1}^{n} (\mathcal{O}^\otimes)^{J} \times_{\mathbb{F}_*^{J}} \{q_i\} \simeq (\mathcal{O}^\otimes)^{J} \times_{\mathbb{F}_*^{J}} \{q\}. \tag{3.33}$$

Proof of the claim. We prove the result for J, the case of I being completely similar. Consider the category $\Delta \downarrow J$ of simplices of J and let ι denote the inclusion $(\Delta \downarrow J)_{\mathrm{nd}} \subset \Delta \downarrow J$ of the full subcategory consisting of all nondegenerate simplices. Writing $\mathcal{F}$ and $\mathcal{F}_i$ for the simplicial presheaves $\mathcal{F}(\sigma) = (\mathcal{O}^\otimes)^{\Delta^n} \times_{\mathbb{F}_*^{\Delta^n}} \{q(\sigma)\}$ and $\mathcal{F}_i(\sigma) = (\mathcal{O}^\otimes)^{\Delta^n} \times_{\mathbb{F}_*^{\Delta^n}} \{q_i(\sigma)\}$ on $\Delta \downarrow J$, the natural equivalences (3.32) give a natural transformation $\gamma\colon \prod_i \mathcal{F} \Rightarrow \mathcal{F}$ which is a levelwise categorical equivalence. Our goal is to show that the equivalence (3.33) is obtained from $\iota^*\gamma\colon \iota^* \prod_i \mathcal{F}_i \Rightarrow \iota^*\mathcal{F}$ by taking the colimit over $(\Delta \downarrow J)_{\mathrm{nd}}$.

First, we show that $\iota^* \prod_i \mathcal{F}_i$ and $\iota^*\mathcal{F}$ are isofibrant diagrams, in the sense of [Lur22, Tag 0349]. We begin by proving that the forgetful functor $\mathcal{U}\colon (\Delta \downarrow J)_{\mathrm{nd}} \to \mathsf{sSet}$ sending $\Delta^n \to J$ to Δ^n is projectively cofibrant (that is cofibrant in the projective global model structure on the diagram category $\mathrm{Fun}((\Delta \downarrow J)_{\mathrm{nd}}, \mathsf{sSet})$). Observe that each simplicial level $\mathcal{U}_k$ of $\mathcal{U}$ decomposes as a coproduct

$$\mathcal{U}_k = \mathcal{U}_k^{\mathrm{nd}} \coprod \mathcal{U}_k^{\mathrm{deg}}$$

of subfunctors of nondegenerate and degenerate simplices respectively. Each of these two subfunctors is a coproduct of representables, so that both are projectively cofibrant. By [Dug01, Corollary 9.4], we deduce that $\mathcal{U}$ is projectively cofibrant. Therefore, the simplicial presheaves $\mathrm{Hom}(\mathcal{U}, \mathcal{O}^\otimes)$ and $\mathrm{Hom}(\mathcal{U}, \mathbb{F}_*)$ on $(\Delta \downarrow J)_{\mathrm{nd}}$ are isofibrant. Taking pullback, it follows that the simplicial presheaf $\mathcal{F}$ is also isofibrant. The same argument proves that $\prod_i \mathcal{F}_i$ is isofibrant.

Since $\iota^*\gamma$ is a levelwise categorical equivalence between isofibrant diagrams, the induced map $\lim(\iota^*\gamma)\colon \lim \prod_i \iota^*\mathcal{F}_i \to \lim \iota^*\mathcal{F}$ on limits is a categorical equivalence. The only remaining step is to identify $\lim \prod_i \iota^*\mathcal{F}_i$ (respectively $\lim \iota^*\mathcal{F}$)

with the left (resp. right) hand side of the second equivalence of (3.33). To this end, note that since J is braced by assumption, the inclusion ι admits a left adjoint and therefore is cofinal. This implies that $J \cong \operatorname{colim}\mathcal{U} \cong \operatorname{colim}\iota^*\mathcal{U}$, from which the result follows easily. □

The proof of lemma 3.5.3 now follows from claim 3.6.1. □

3.6.3 Proof of lemma 3.5.5

Proof of lemma 3.5.5. Recall that the morphism ι is the inclusion

$$\iota\colon \widetilde{K} = \left(\Delta^{013467} \bigcup_{\Delta^{0347}} \Delta^{023457}\right) \bigcup_{\Delta^{013} \underset{\Delta^{03}}{\cup} \Delta^{023}} \Delta^{0123} \longrightarrow \Delta^7$$

using the numbering introduced in diagram (3.22). First, note that ι factors through $\hat{K} = \widetilde{K} \underset{\Lambda^{567}_7}{\cup} \Delta^{567}$ and the right anodyne inclusion $\widetilde{K} \to \hat{K}$ satisfies the conditions of the statement. It therefore suffices to show that the induced map $\hat{K} \to \Delta^7$ is inner anodyne. Note that the spine inclusion $\mathrm{Sp}^7 \to \Delta^7$, which is inner anodyne, factors through $\hat{K}$. By the right cancellation property for inner anodyne morphisms, it suffices to show that $\mathrm{Sp}^7 \to \hat{K}$ is inner anodyne. We decompose the latter inclusion in several steps; first, it is easy to see that the two morphisms

$$\mathrm{Sp}^7 \to \mathrm{Sp}^7 \underset{\mathrm{Sp}^{0123}}{\cup} \Delta^{0123} \to \left(\mathrm{Sp}^7 \underset{\mathrm{Sp}^{0123}}{\cup} \Delta^{0123}\right) \underset{\mathrm{Sp}^{4567}}{\cup} \Lambda^{4567}_7 =: S$$

are inner anodyne. As the inclusion $\mathrm{Sp}^7 \to \Delta^7$ factors through the composite map $\mathrm{Sp}^7 \to S$, the remaining steps consists in adding to S the simplices Δ^{013467} and Δ^{023457}. One verifies that the intersection between S and Δ^{013467} is given by

$$S \cap \Delta^{013467} = \Delta^{013} \underset{\Delta^{\{3\}}}{\cup} \Delta^{34} \underset{\Delta^{\{4\}}}{\cup} \Delta^{467}$$

so that the inner anodyne inclusion $\mathrm{Sp}^{013467} \to \Delta^7$ factors through $S \cap \Delta^{013467}$ as an inner anodyne map. Therefore, by right cancellation, we deduce that $S \cap \Delta^{013467} \to \Delta^{013467}$ is inner anodyne. Since the inclusion of S into the simplicial set $\hat{S} = S \underset{S\cap\Delta^{013467}}{\cup} \Delta^{013467}$ is the pushout of the inclusion $S \cap \Delta^{013467} \to \Delta^{013467}$, it is also inner anodyne. Finally, we will prove that the same holds for the inclusion $\hat{S} \to \hat{S} \cup \Delta^{023567} = \hat{K}$. We proceed as before: the intersection of $\hat{S}$ and Δ^{023457} is given by

$$\hat{S} \cap \Delta^{023457} = \Delta^{023} \underset{\Delta^{\{3\}}}{\cup} \Delta^{34} \underset{\Delta^{\{4\}}}{\cup} \Delta^{457}$$

and the inclusion $\mathrm{Sp}^{023457} \to \Delta^{023457}$ factors through it. To prove that $\hat{S} \to \hat{K}$ is inner anodyne, it is then enough to show that $\mathrm{Sp}^{023457} \to \hat{S} \cap \Delta^{023457}$ has this property, which is obvious. □

Step	1-simplices	2-simplices	3-simplices	4-simp.	5-simp.	6-simp.	7-simp.
Initial step $\widetilde{K}$	01 02 03 04 05 06 07 12 13 14 16 17 23 24 25 27 34 35 36 37 45 46 47 57 67	012 013 014 016 017 023 024 025 027 034 035 036 037 045 046 047 057 067 123 134 136 137 146 147 167 234 235 237 245 247 257 345 346 347 357 367 457 467	0123 0134 0136 0137 0146 0147 0167 0234 0235 0237 0245 0247 0257 0345 0346 0347 0357 0367 0457 0467 1346 1347 1367 1467 2345 2347 2357 2457 3457 3467	01346 01347 01367 01467 02345 02347 02357 02457 03457 03467 13467 23457	013467 023457		
ι_1	15	125					
ι_2	26	236					
ι_3	56	567					
ι_4		015	0125				
ι_5		026	0236				
ι_6		124	1234				
ι_7		126	1236				
ι_8		135	1235				
ι_9		145	1245				
ι_{10}		456	4567				
ι_{11}		356	3567				
ι_{12}		267	2367				
ι_{13}		256	2567				
ι_{14}		267	2467				
ι_{15}		156	1256				
ι_{16}		056	0156				
ι_{17}		157	1457				
ι_{18}		127	1257				
ι_{19}			0124	01234			
ι_{20}			0135	01235			
ι_{21}			0126	01236			
ι_{22}			0145	01245			
ι_{23}			0256	01256			
ι_{24}			0157	01457			
ι_{25}			0127	01257			
ι_{26}			0267	02367			
ι_{27}			1247	12457			
ι_{28}			1237	12347			
ι_{29}			1267	12367			
ι_{30}			1357	12357			
ι_{31}			1567	12567			
ι_{32}			0567	01567			
ι_{33}			3456	34567			
ι_{34}			2456	24567			
ι_{35}			2356	23567			
ι_{36}			2346	23467			
ι_{37}			1456	14567			
ι_{38}			1356	13567			
ι_{39}			1345	13457			
ι_{40}			1246	12467			
ι_{41}			0456	04567			
ι_{42}			0356	03567			
ι_{43}			0246	02467			
ι_{44}				01247	012457		
ι_{45}				12345	123457		
ι_{46}				12346	123467		
ι_{47}				12356	123567		
ι_{48}				12456	124567		
ι_{49}				13456	134567		
ι_{50}				23456	234567		
ι_{51}				01345	012345		
ι_{52}				03456	034567		
ι_{53}				01237	012347		
ι_{54}				01357	012357		
ι_{55}				01267	012367		
ι_{56}				02567	012567		
ι_{57}				02456	024567		
ι_{58}				02356	023567		
ι_{59}				02346	023467		
ι_{60}				01456	014567		
ι_{61}				01356	013567		
ι_{62}				01246	012346		
ι_{63}					013457	0123457	
ι_{64}					012467	0123467	
ι_{65}					123456	1234567	
ι_{66}					023456	0234567	
ι_{67}					013456	0134567	
ι_{68}					012456	0124567	
ι_{69}					012356	0123567	
ι_{70}						0123456	01234567

Figure 3.2: Iterative construction of the non-degenerate simplices of Δ^7 using right horn inclusion, starting from $\widetilde{K}$.

Alternative proof of lemma 3.5.5. For the record, we provide another proof of lemma 3.5.5 by exhibiting an explicit decomposition of ι as a composite of marked anodyne morphisms.

The first map of the composition is the canonical morphism

$$\iota_1 \colon \widetilde{K} =: \widetilde{K}_1 \longrightarrow \widetilde{K}_2 := \left(\widetilde{K} \cup_{\Lambda_2^{125}} \Delta^{125}\right)$$

induced by the inner horn inclusion $\Lambda_2^{125} \subset \Delta^{125}$; it corresponds to adding to the diagram $\widetilde{K} \to \mathcal{E}$ a 2-simplex 125 and a 1-simplex 15. The next morphism is $\iota_2 \colon \widetilde{K}_2 \to \widetilde{K}_2 \cup_{\Lambda_3^{236}} \Delta^{236}$.

For a more systematic presentation of the next morphisms ι_i, we use figure 3.2. In this table, the initial setup is the list of the non-degenerate simplices of Δ^7 that belong to $\widetilde{K}$. At step ι_i, the line of the table contains exactly two non-degenerate simplices of Δ^7, which are of the form $J_i \setminus \{k_i\}$ and J_i, representing the horn inclusion $\Lambda_{k_i}^{J_i} \hookrightarrow \Delta^{J_i}$ of which ι_i is a pushout.

To verify that the construction is valid, one has to check that each of the non-degenerate simplices of positive dimension of Δ^7 appears exactly once in the above table, and moreover that at each step of the construction, all the faces of the horn $\Lambda_{k_i}^{J_i}$ have already been constructed. For instance, in the step ι_2 corresponding to the horn inclusion $\Lambda_3^{236} \hookrightarrow \Delta^{236}$, one has to verify that the 2-simplices 23 and 36 have already been built (as well as all non-degenerate simplices of Λ_3^{236} of dimension less than 2) and that neither 23 nor 236 have. Verifying similarly the other steps is a simple but tedious exercise. $\square$

Chapter 4

Comparison with Lurie's spaces of extensions

Contents

4.1 Statement of the results

We turn to problem B of the introduction, namely the question of the comparison between Mann–Robalo's and Lurie's models for spaces of extensions. We provide a solution to this problem through theorem 4.1.1.

More precisely, let $\mathcal{O}^\otimes$ be a unital ∞-operad and fix an active morphism σ, viewed as an object in the twisted arrow ∞-category $\mathcal{T} := \mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^\otimes$ of the monoidal envelope of $\mathcal{O}^\otimes$.

The goal of the present chapter is to prove the equivalence between the fiber $\mathcal{BO}_\sigma$ of the brane fibration $\mathcal{BO} \to \mathcal{T}$ (introduced in definition 2.4.1) at the operation σ coincide with the ∞-category of extensions $\mathrm{Ext}(\sigma)$ of σ, thereby proving the corresponding claim in [MR18] [1]. Our method consist in providing an explicit zigzag of categorical equivalences between $\mathcal{BO}_\sigma$ and $\mathrm{Ext}(\sigma)$.

Theorem 4.1.1 (Theorem B)**.** *Let σ be an active morphism in a unital ∞-operad $\mathcal{O}^\otimes$. Then the fiber $\mathcal{BO}_\sigma$ of the brane fibration and the ∞-category of extensions $\mathrm{Ext}(\sigma)$ are equivalent in Cat_∞.*

Corollary 4.1.2. *Let σ be an active morphism in a unital ∞-operad $\mathcal{O}^\otimes$, whose underlying ∞-category is an ∞-groupoid. Then $\mathcal{BO}_\sigma$ and $\mathrm{Ext}(\sigma)$ are equivalent Kan complexes.*

The difference between the simplicial sets $\mathrm{Ext}(\sigma)$ and $\mathcal{BO}_\sigma$ can be observed from their set of 0-simplices. Both sets parametrize diagrams in $\mathcal{O}^\otimes$, respectively of the form

$$\begin{tikzcd} \bullet \arrow[r, "\sigma"] \arrow[d, hook] \arrow[dr] & \bullet \arrow[d, "\wr"] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \bullet \arrow[r, "\sigma"] \arrow[d, hook] \arrow[dr] & \bullet \\ \bullet \arrow[r] \arrow[ur] & \bullet, \arrow[u, "\wr"'] \end{tikzcd} \tag{4.1}$$

that is, of respective shape $\Delta^1 \times \Delta^1$ and Δ^3, such that the following conditions are satisfied:

- the top horizontal arrow is sent to σ;
- the left vertical arrow is sent to an atomic map;
- the right vertical arrow is sent to an equivalence in $\mathcal{O}^\otimes$.

Informally, since the right vertical arrow is marked as an equivalence, both diagrams in (4.1) encode the same data, namely that of a triangle of the form

$$\begin{tikzcd} \bullet \arrow[r, "\sigma"] \arrow[d, hook] & \bullet \\ \bullet \arrow[ur] & \end{tikzcd} \tag{4.2}$$

Our proof follows this idea and relies on finding suitable generalizations of the previous diagram for higher dimensional simplices in $\mathrm{Ext}(\sigma)$ and $\mathcal{BO}_\sigma$.

4.2 Definition of spaces interpolating between $\mathcal{BO}_\sigma$ and $\mathrm{Ext}(\sigma)$

We first discuss the shape of diagrams represented by general simplices of $\mathrm{Ext}(\sigma)$ and $\mathcal{BO}_\sigma$. Let K be a simplicial set.

To help the reader with the cumbersome definitions and notations, we recommend to look at figures 4.2.3 and 4.2.3 where the different diagrams introduced in this section are depicted, for the cases $K = \Delta^0$ and $K = \Delta^1$.

[1] More precisely, for the claim $\mathcal{BO}_\sigma \simeq \mathrm{Ext}(\sigma)$ to be correct, one needs to adopt definition 2.2.3 as one's definition of the ∞-category of extensions, instead of [Lur17, Definition 3.3.1.4.] used in [MR18]. This minor change is argued in remark 2.2.5.

4.2.1 Diagrams indexed by $\mathrm{Ext}(\sigma)$

Let $F_0(K)$ denote the simplicial set $K^{\triangleleft} \times \Delta^1$ and consider the canonical projection $F_0(K) \to \Delta^1$. The vertices in the fiber over 0 will be denoted as x, whereas those in the fiber over 1 will be written $\overline{x}$, where $x \in K^{\triangleleft}$. Unravelling definition 2.2.3, we see that morphisms $K \to \mathrm{Ext}(\sigma)$ are identified with those diagrams $\alpha\colon F_0(K) \to \mathcal{O}^{\otimes}$ that send all morphisms in $K \times \{1\} \subset K^{\triangleleft} \times \Delta^1$ to equivalences and moreover satisfy the following condition.

Condition $(\star)_{0,\sigma}$: for every vertex $k \in K$,

- the morphism $\triangleleft \to k$ in $F_0(K)$ is sent to an atomic map;
- the morphism $k \to \overline{k}$ in $F_0(K)$ is sent to an active map;
- the restriction of α to $\{\triangleleft\} \times \Delta^1$ is σ; and
- for every morphism $k_0 \to k_1$ in K, the corresponding morphism in $F_0(K)$ is compatible with extensions.

Letting $F_0^+(K)$ denote the marked simplicial set obtained from $F_0(K)$ by further marking the 1-simplices of $K^{\triangleleft} \times \{1\}$, one obtains a bijection between $\mathrm{Hom}(K, \mathrm{Ext}(\sigma))$ and a subset $\mathrm{Hom}^{\sigma}(F_0^+(K), \mathcal{O}^{\otimes})$ of $\mathrm{Hom}_{\mathsf{sSet}^+}(F_0^+(K), \mathcal{O}^{\otimes,\natural})$ given by those diagrams that satisfy condition $(\star)_{0,\sigma}$.

4.2.2 Diagrams indexed by $\mathcal{BO}_\sigma$

Observe that the set of diagrams from K to $\mathcal{BO}_\sigma$ is a certain subset of

$$\mathrm{Hom}(K, \mathcal{T}^{\Delta^1} \times_{\mathcal{T}} \{\sigma\}) = \mathrm{Hom}(K \times \Delta^1, \mathcal{T}) \underset{\mathrm{Hom}(K,\mathcal{T})}{\times} \{\sigma \circ \mathrm{proj}_K\}.$$

Using remark 2.4.2, we may then identify $\mathrm{Hom}(K, \mathcal{BO}_\sigma)$ with a particular subset of $\mathrm{Hom}(G(K), \mathcal{O}^{\otimes})$, where $G(K)$ denotes the pushout

$$G(K) := s_*(K \times \Delta^1) \underset{s_*(K\times\{0\})}{\amalg} s_*\{0\}.$$

Let $G^+(K)$ denote the marked simplicial set obtained from $G(K)^\flat$ by further marking the arrows of the form $\overline{(k,1)} \to \overline{(k,0)}$, for all vertices k in K. Unraveling the definition of $\mathcal{BO}$, we will identify $\mathrm{Hom}(K, \mathcal{BO}_\sigma)$ with the subset $\mathrm{Hom}^{\sigma}(G^+(K), \mathcal{O}^{\otimes,\natural})$ of $\mathrm{Hom}_{\mathsf{sSet}^+}(G^+(K), \mathcal{O}^{\otimes,\natural})$ consisting of those diagrams verifying the following condition:

Condition $(\star)_{G,\sigma}$**:**

- the arrow $s_*\{0\} \cong \Delta^1$ in $G^+(K)$ is sent to σ;
- for every $k \in K$, the 3-simplex $s_*(\{k\} \times \Delta^1)$ given by the vertices $(k,0)(k,1)\overline{(k,1)(k,0)}$ is sent to an object of $\mathcal{BO}$;
- for every 1-simplex $f\colon k \to k'$ in K, the corresponding diagram $s_*(f \times \Delta^1)$ is sent to a morphism of $\mathcal{BO}$; more explicitly, this conditions means that the arrow $f \times \{1\}\colon (k,1) \to (k',1)$ in $G^+(K)$ is sent to a map in $\mathcal{O}^{\otimes}$ that is compatible with extensions (in the sense of definition 2.4.1).

4.2.3 Definition of intermediate steps

We will relate $\mathrm{Ext}(\sigma)$ and $\mathcal{BO}_\sigma$ using intermediate ∞-categories $\mathcal{C}_i$, for $i \in \{1,2,3\}$, whose construction is given by the following procedure. For K a simplicial set, the set $\mathrm{Hom}(K, \mathcal{C}_i)$ is identified with the subset $\mathrm{Hom}^\sigma(F_i^+(K), \mathcal{O}^{\otimes,\natural})$ of $\mathrm{Hom}_{\mathsf{sSet}^+}(F_i^+(K), \mathcal{O}^{\otimes,\natural})$ consisting of all diagrams $F_i^+(K) \to \mathcal{O}^{\otimes,\natural}$ that satisfy a certain condition denoted $(\star)_{i,\sigma}$. Here, the marked simplicial sets $F_i^+(K)$ are to be thought of as shapes which interpolate between $F_0^+(K)$ and $G^+(K)$. These marked simplicial sets will fit into a zigzag of the form

$$F_0^+(K) \xrightarrow{i_0(K)} F_1^+(K) \xleftarrow{i_1(K)} F_2^+(K) \xrightarrow{i_2(K)} F_3^+(K) \xleftarrow{p(K)} G^+(K), \tag{4.3}$$

natural in K, which yields a zigzag of functors of ∞-categories

$$\mathrm{Ext}(\sigma) \xleftarrow{i_0^*} \mathcal{C}_1 \xrightarrow{i_1^*} \mathcal{C}_2 \xleftarrow{i_2^*} \mathcal{C}_3 \xrightarrow{p^*} \mathcal{BO}_\sigma. \tag{4.4}$$

Notation 4.2.1. Unless ambiguous, we will write i_n instead of $i_n(K)$.

We now describe the different marked simplicial sets $F_i^+(K)$, their associated condition $(\star)_{i,\sigma}$ and the morphisms i_0, i_1, i_2 and p.

(F_1^+) The marked simplicial set $F_1^+(K)$ is obtained from the simplicial set

$$F_1(K) = \operatorname*{colim}_{\Delta^m \to K^\triangleleft} (\Delta^m * \Delta^m),$$

by marking all edges of the second copy of Δ^m.

(i_0) Using that $F_0(K)$ can be rewritten as $\operatorname*{colim}_{\Delta^m \to K^\triangleleft}(\Delta^m \times \Delta^1)$, the canonical inclusions $\Delta^m \times \Delta^1 \to \Delta^m * \Delta^m$ induce a map of marked simplicial sets $i_0\colon F_0^+(K) \to F_0^+(K)$. Similarly to the case of $F_0(K)$, we label vertices of $F_1(K)$ as x or $\overline{x}$ using the obvious projection $F_1(K) \to \Delta^1$, where $x \in K^\triangleleft$.

(F_2^+) The marked simplicial set $F_2^+(K)$ is defined as $F_2(K)^\flat$, where

$$F_2(K) = K^{\triangleleft\triangleright}.$$

(i_1) Using the isomorphism $F_2(K) \cong \operatorname*{colim}_{\Delta^m \to K^\triangleleft} (\Delta^m)^\triangleright$, we define the inclusion $i_1\colon F_2^+(K) \to F_1^+(K)$ as induced by the morphisms $i_{1,m}\colon (\Delta^m)^\triangleright \to F_1^+(K)$ sending Δ^m to the first copy of itself in $\Delta^m * \Delta^m$ and $\triangleright$ to $\overline{\triangleleft}$.

(F_3^+) The marked simplicial set $F_3^+(K)$ is obtained from the simplicial set

$$F_3(K) = s_*(K^\triangleleft) \cong \operatorname*{colim}_{\Delta^m \to K^\triangleleft} (\Delta^m * \Delta^{m,\mathrm{op}})$$

by marking all edges of the second copy $\Delta^{m,\mathrm{op}}$.

(i_2) The morphism $i_2\colon F_2^+(K) \to F_3^+(K)$ is induced by the inclusions $(\Delta^m)^\triangleright \to F_3^+(K)$, that send $\triangleright$ to $\overline{\triangleleft} \in F_3^+(K)$ and Δ^m to the first copy of itself in $\Delta^m * \Delta^{m,\mathrm{op}}$.

(p) We now define the morphism $p\colon G^+(K) \to F_3^+(K)$. We will make use of the canonical isomorphism

$$s_*(K \times \Delta^1) \cong \operatorname*{colim}_{\Delta^n \to K} s_*(\Delta^n \times \Delta^1).$$

Given an n-simplex $\tau_n\colon \Delta^n \to K$ of K, consider the projection $\tilde{p}_n\colon \Delta^n \times \Delta^1 \to (\Delta^n)^{\triangleleft}$ that sends the vertex $(k,1)$ to k and all vertices of the form $(i,0)$ to $\triangleleft$. The composition $\tau_n \circ \tilde{p}_n\colon \Delta^n \times \Delta^1 \to K^{\triangleleft}$ yields a map

$$s_*\tilde{p}_n\colon s_*(\Delta^n \times \Delta^1) \to s_*\left((\Delta^n)^{\triangleleft}\right) \subseteq \operatorname*{colim}_{\Delta^m \to K^{\triangleleft}} s_*(\Delta^m) = F_3(K).$$

For varying n-simplices τ_n, the maps $s_*\tilde{p}_n$ organize into a map $\tilde{p}\colon s_*(K \times \Delta^1) \to F_3^+(K)$, whose restriction to $s_*(K \times \{0\})$ factors through $s_*\{0\}$ as the inclusion $s_*\{0\} \cong \Delta^{\triangleleft\overline{\triangleleft}} \subseteq F_3^+(K)$ and therefore induces our map $p\colon G(K) \to F_3(K)$.

$(\star)_{n,\sigma}$ The conditions $(\star)_{n,\sigma}$ for $n \in \{1,2,3\}$ are given *mutatis mutandis* by condition $(\star)_{0,\sigma}$.

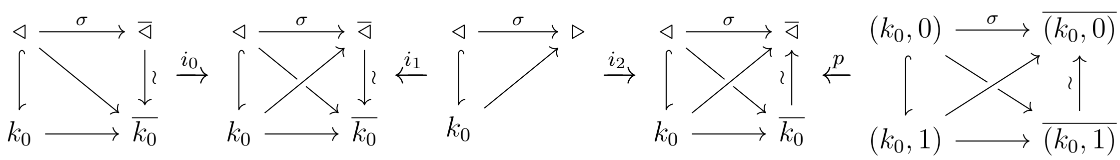


Figure 4.1: Zigzag diagram (4.3) for $K = \{k_0\} \cong \Delta^0$

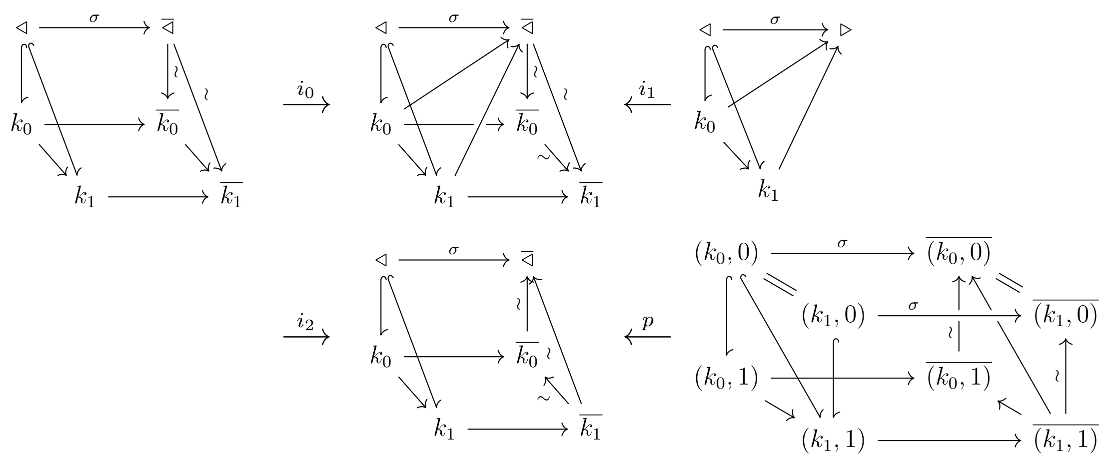


Figure 4.2: Zigzag diagram (4.3) for $K = \{k_0 \to k_1\} \cong \Delta^1$

Lemma 4.2.2. *Let $n \in \{1, 2, 3\}$ and let K be a simplicial set. Consider the corresponding morphism $i_n\colon F_a^+(K) \to F_b^+(K)$, where the indices a and b are determined by n. Then a functor $\alpha\colon F_a^+(K) \to \mathcal{O}^{\otimes,\natural}$ satisfies condition $(\star)_{a,\sigma}$ if and only if $i_n(K)^*(\alpha)$ satisfies condition $(\star)_{b,\sigma}$.*

Proof. The lemma follows from inspection of the different conditions. $\square$

We now establish that the definition of $(\mathcal{C}_n)_\bullet$ as $\mathrm{Hom}^\sigma(F_n^+(\Delta^\bullet), \mathcal{O}^{\otimes,\natural})$ yields an ∞-category.

Lemma 4.2.3. *For $n \in \{1, 2, 3\}$, the simplicial set $\mathcal{C}_n$ is an ∞-category.*

Proof. We first prove the result for the simplicial set $\mathcal{C}_1$. Let $k, m \in \mathbb{N}$ with $0 < k < m$ and $f \in \mathrm{Hom}^\sigma(F_1^+(\Lambda_k^m), \mathcal{O}^{\otimes,\natural})$. We will show that the existence of a lift in the following diagram

$$\begin{tikzcd}
F_1^+(\Lambda_k^m) \arrow[r, "f"] \arrow[d] & \mathcal{O}^{\otimes,\natural} \\
F_1^+(\Delta^m) \arrow[ur, dashed] &
\end{tikzcd}$$

by decomposing the vertical map as a sequence of inner anodyne morphisms. First, let X_0 denote the simplicial set $\Delta^{\triangleleft 0\ldots n} \cup F_1^+(\Lambda_k^m) \cup \Delta^{\overline{\triangleleft 0}\ldots\overline{m}}$. It is clear that the inclusion $F_1^+(\Lambda_k^m) \to X_0$ is inner anodyne. Since $\emptyset \to \Delta^m$ is both right and left anodyne, we may choose an increasing filtration

$$\emptyset = \Lambda_0 \subset \Lambda_1 \subset \cdots \subset \Lambda_r = \Delta^m \tag{4.5}$$

of subsimplicial sets of Δ^m such that each inclusion $\Lambda_j \to \Lambda_{j+1}$ is the pushout of a horn inclusion $\Lambda_{k_j}^{m_j} \to \Delta^{m_j}$, which either is inner anodyne or satisfies $m_j \leqslant 1$. Note that in the latter case, the horn inclusion is left or right anodyne. Introduce the simplicial sets

$$\begin{aligned}
X_j &= X_0 \cup (\Lambda_j^\triangleleft * \overline{\Lambda_j^\triangleleft}) \\
Y_j &= X_0 \cup (\Lambda_{j+1}^\triangleleft * \overline{\Lambda_j^\triangleleft}) \cup (\Lambda_j^\triangleleft * \overline{\Lambda_{j+1}^\triangleleft}),
\end{aligned}$$

so that the inclusion $X_0 \to F_1^+(\Delta^m)$ can be written as the sequence of inclusions

$$X_0 \subseteq Y_0 \subseteq X_1 \subseteq Y_1 \subseteq \ldots \subseteq X_{r-1} \subseteq Y_{r-1} \subseteq X_r = F_1^+(\Delta^m).$$

Each inclusion $Y_j \to X_{j+1}$ is the pushout of the morphism $(\Lambda_j^\triangleleft \subseteq \Lambda_{j+1}^\triangleleft) \circledast (\overline{\Lambda_j^\triangleleft} \subseteq \overline{\Lambda_{j+1}^\triangleleft})$; lemma A.1.9 implies that it is inner anodyne. For $p \in [m]$, let $\Lambda_j(p)$ denote the intersection of Λ_j with the face of Δ^m opposed to vertex p. For every j with $0 \leqslant j < r$, let p_j denote the index of the unique face of Δ^m that contains $\Lambda_{j+1} \setminus \Lambda_j$. Now consider the inclusion $X_j \to Y_j$. By construction of the filtration (4.5), this inclusion is obtained as a pushout of the map

$$\begin{gathered}
\left(\Lambda_j^\triangleleft * \overline{\Lambda_j^\triangleleft(p_j)} \cup \Lambda_j^\triangleleft(p_j) * \overline{\Lambda_{j+1}^\triangleleft(p_j)}\right) \cup \left(\Lambda_j^\triangleleft(p_j) * \overline{\Lambda_j^\triangleleft} \cup \Lambda_{j+1}^\triangleleft(p_j) * \overline{\Lambda_j^\triangleleft(p_j)}\right) \\
\downarrow \\
\left(\Lambda_j^\triangleleft * \overline{\Lambda_{j+1}^\triangleleft(p_j)}\right) \cup \left(\Lambda_{j+1}^\triangleleft(p_j) * \overline{\Lambda_j^\triangleleft}\right).
\end{gathered}$$

This last map is the pushout of the map $(\Lambda^{\triangleleft}_j(p_j) \subseteq \Lambda^{\triangleleft}_j) \boxtimes (\Lambda^{\triangleleft}_j(p_j) \subseteq \Lambda^{\triangleleft}_{j+1}(p_j))$ with its symetric; it is therefore an inner anodyne map by lemma A.1.9, as desired. Therefore f extends to a map $\tilde{f}\colon F_1^+(\Delta^m) \to \mathcal{O}^{\otimes,\natural}$. The fact that $\tilde{f}$ belongs to the subset $\mathrm{Hom}^\sigma(F_1^+(\Delta^m), \mathcal{O}^{\otimes,\natural})$ follows directly from the similar hypothesis for f; this concludes the proof that $\mathcal{C}_1$ is an ∞-category.

One can give a very similar proof for $\mathcal{C}_3$, simply reversing the direction of the edges in $\overline{\Delta^m}$; we shall therefore omit the details.

We now turn to the case of $\mathcal{C}_2$. As before, let $k, m \in \mathbb{N}$ with $0 < k < m$. It is enough to prove that the inclusion $F_2^+(\Lambda^m_k) \to F_2^+(\Delta^m)$ is inner anodyne. One simply observes that this inclusion can be described as successively adding to $F_2^+(\Lambda^m_k) = (\Lambda^m_k)^{\triangleleft\triangleright}$ fillers of the inner horns $\Lambda^{0\ldots m}_k$, $\Lambda^{\triangleleft 0\ldots m}_k$, $\Lambda^{0\ldots m\triangleright}_k$ and $\Lambda^{\triangleleft 0\ldots m\triangleright}_k$, which proves the claim. □

4.3 Proof of theorem 4.1.1

4.3.1 Strategy of proof

We explain our approach to proving that the functors i_0^*, i_1^* and i_2^*

$$\mathrm{Ext}(\sigma) \xleftarrow{i_0^*} \mathcal{C}_1 \xrightarrow{i_1^*} \mathcal{C}_2 \xleftarrow{i_2^*} \mathcal{C}_3$$

from zigzag (4.4) are all equivalences of ∞-categories. The remaining case of the functor $p^*\colon \mathcal{C}_3 \to \mathcal{B}\mathcal{O}_\sigma$ is treated separately, with different arguments, in section 4.3.4.

Fix $n \in \{1,2,3\}$ and consider the associated natural transformation $i_n\colon F_a^+ \to F_b^+$, where the indices $a, b \in \{0,1,2,3\}$ are determined by n.

Notation 4.3.1. Let $\mathcal{J}$ denote the nerve of free-living isomorphism (i.e. the contractible groupoid on 2 objects), which is an interval object for Joyal's model structure (see the book by Cisinski [Cis19] and also [Cam21, Appendix A]). Given a simplicial set K, we introduce the pushout

$$F^+_{a,b,\mathcal{J}}(K) = F_a^+(K \times \mathcal{J}) \coprod_{F_a^+(K)^{\amalg 2}} F_b^+(K)^{\amalg 2}.$$

Note that $i_n(\overline{K})$ factors as a composite

$$F_a^+(K \times \mathcal{J}) \xrightarrow{j_n} F^+_{a,b,\mathcal{J}}(K) \xrightarrow{k_n} F_b^+(K \times \mathcal{J}). \tag{4.6}$$

The following result is central in our approach.

Lemma 4.3.2. *Let $n \in \{1,2,3\}$. Suppose that for all simplicial sets K, the morphism $i_n(K)\colon F_a^+(K) \to F_b^+(K)$ is marked anodyne and the induced morphism $k_n\colon F^+_{a,b,\mathcal{J}}(K) \to F_b^+(K \times \mathcal{J})$ from factorization (4.6) is a monomorphism which is bijective on 0-simplices. Then the induced map*

$$i_n(K)^*\colon \pi_0 \,\mathrm{Map}(K, \mathcal{C}_b) \to \pi_0 \,\mathrm{Map}(K, \mathcal{C}_a)$$

is a bijection for all K.

Proof. Using the left lifting property of marked anodyne morphisms against morphisms of the form $X^\natural \to *$ for X an ∞-category (lemma A.1.6), we obtain that the map

$$\mathrm{Hom}_{\mathsf{sSet}^+}(F_b^+(K), \mathcal{O}^{\otimes,\natural}) \to \mathrm{Hom}_{\mathsf{sSet}^+}(F_a^+(K), \mathcal{O}^{\otimes,\natural})$$

is surjective. Hence so is its quotient

$$i_n(K)^*\colon \pi_0 \operatorname{Map}(K, \mathcal{C}_b) \to \pi_0 \operatorname{Map}(K, \mathcal{C}_a).$$

We now prove that $i_n(K)^*$ is also injective. Let $\alpha, \alpha'\colon F_b^+(K) \to \mathcal{O}^{\otimes,\natural}$ be such that $i_n^*(\alpha) \simeq i_n^*(\alpha')$ in $\operatorname{Map}(K, \mathcal{C}_n)$. Using the standard categorical cylinder $K \amalg K \to K \times \mathcal{J} \to K$ of K, the latter condition means that we can fill the following diagram of solid lines

$$\begin{tikzcd}
F_a^+(K) \arrow[dr] \arrow[drr, bend left, "i_n^*(\alpha)"] & & \\
& F_a^+(K \times \mathcal{J}) \arrow[r, dotted, "\overline{\alpha}"] & \mathcal{O}^{\otimes,\natural}. \\
F_a^+(K) \arrow[ur] \arrow[urr, bend right, "i_n^*(\alpha')"'] & &
\end{tikzcd} \tag{4.7}$$

so that $\overline{\alpha}$ respects conditions $(*)_{a,\sigma}$. To show that $\alpha \simeq \alpha'$ in $\operatorname{Map}(K, \mathcal{C}_b)$, we have to prove that the corresponding diagram for F_b^+, namely

$$\begin{tikzcd}
F_b^+(K) \arrow[dr] \arrow[drr, bend left, "\alpha"] & & \\
& F_b^+(K \times \mathcal{J}) \arrow[r, dotted, "\tilde{\alpha}"] & \mathcal{O}^{\otimes,\natural}, \\
F_b^+(K) \arrow[ur] \arrow[urr, bend right, "\alpha'"'] & &
\end{tikzcd} \tag{4.8}$$

can be filled by a map $\tilde{\alpha}\colon F_b^+(K) \to \mathcal{O}^{\otimes,\natural}$ that satisfies conditions $(*)_{b,\sigma}$. Now observe that $i_n(K \times \mathcal{J})$ factors through the pushout $F^+_{a,b,\mathcal{J}}(K)$. By hypothesis, $i_n(K)$ is marked anodyne, whence so is j_n. Using the right simplification property of marked anodyne morphisms (Proposition A.1.7), we deduce that there exists a lift $\tilde{\alpha}$ in the following diagram

$$\begin{tikzcd}
F_a^+(K \times \mathcal{J}) \arrow[d, "j_n"] \arrow[dr, bend left, "\overline{\alpha}"] \arrow[dd, bend right=60, "i_n"'] & \\
F^+_{a,b,\mathcal{J}}(K) \arrow[r, "(\alpha \amalg \alpha', \overline{\alpha})"] \arrow[d, "k_n"] & \mathcal{O}^{\otimes,\natural}. \\
F_b^+(K \times \mathcal{J}) \arrow[ur, dotted, bend right, "\widetilde{\alpha}"'] &
\end{tikzcd} \tag{4.9}$$

By lemma 4.2.2, any lift $\tilde{\alpha}$ as above will automatically satisfies conditions $(*)_{b,\sigma}$. This shows that α and α' are equivalent as points in $\mathrm{Map}(K, \mathcal{C}_b)$, as desired. □

We will prove that the maps i_0, i_1 and i_2 are marked anodyne and use the above lemma to deduce that the ∞-categories $\mathrm{Ext}(\sigma)$, $\mathcal{C}_1$, $\mathcal{C}_2$ and $\mathcal{C}_3$ are equivalent. The remaining step will be to deal with the map p, which is a marked equivalence but not a marked anodyne morphism, so that we cannot use lemma 4.3.2; we will therefore rely on a different argument, involving a careful analysis of categorical cylinders in $\mathcal{C}_3$ and $\mathcal{B}\mathcal{O}_\sigma$.

4.3.2 Study of the map i_0

Lemma 4.3.3. *For every simplicial set K, the morphism $k_0(K)\colon F^+_{0,1,\mathcal{J}}(K) \to F^+_1(K)$ is a monomorphism which is bijective on 0-simplices.*

Proof. Let $K \in \mathsf{sSet}$. Observe that $i_0(K)\colon F^+_0(K) \to F^+_1(K)$ is a bijection on 0-simplices. Since K is arbitrary, the map $i_0(K \times \mathcal{J})$ also has this property. Moreover, so does the map $j_0(K)\colon F^+_0(K \times \mathcal{J}) \to F^+_{0,1,\mathcal{J}}(K)$, as it is obtained as a cobase change of $i_0(K)^{\amalg 2}$. By 2-out-of-3 property, we deduce that $k_0(K)$ also has this property.

Let us show that $k_0(K)$ is a monomorphism. Suppose that t and t' are two m-simplices of $F^+_{0,1,\mathcal{J}}(K)$ with the same image under $k_0(K)$. We separate three cases:

(1) either both t and t' lift to simplices of $F^+_0(K \times \mathcal{J})$, or

(2) both lift to $F^+_1(K)^{\amalg 2}$, or

(3) t lifts to a simplex in $F^+_0(K \times \mathcal{J})$ and t' to one in $F^+_1(K)^{\amalg 2}$.

For the first case, note that $j_0(K\times\mathcal{J})$ is defined as a pushout of a monomorphism, hence is a monormophism, so that $t = t'$ as desired. For the second case, writing $\tilde{t}$ and $\tilde{t}'$ for choices of lifts of t and t' in $F^+_1(K)^{\amalg 2}$, we easily see that either $\tilde{t}$ and $\tilde{t}'$ belong to the same of the two copies of $F^+_1(K)$, or they factor through $(\Delta^{\triangleleft\triangleleft})^{\amalg 2}$. Finally, in the third case, the simplices t and t' have to factor through one of the two copies of $F^+_0(K)$; since $F^+_0(K) \to F^+_1(K \times \mathcal{J})$ is a monomorphism, the simplices t and t' must coincide. □

Proposition 4.3.4. *For every simplicial set K, the map $i_0\colon F^+_0(K) \to F^+_1(K)$ is marked anodyne.*

Proof. Recall the marking on these simplicial sets: on $F^+_0(K) = K^\triangleleft \times \Delta^1$, the marked edges are those of the form $\triangleleft \to \overline{y}$, with $y \in K$, whereas for $F^+_1(K) = \operatorname*{colim}_{\Delta^m \to K^\triangleleft}(\Delta^m * \Delta^m)$, all edges $\overline{x} \to \overline{y}$ with $x \to y$ in $K^\triangleleft$ are marked. The morphism i_0 factors through the marked simplicial set $\tilde{F}^+_0(K)$ obtained from $F^+_0(K)$ by further marking the edges $(x, 1) \to (y, 1)$, for $x \to y$ in $K^\triangleleft$. The resulting inclusion $F^+_0(K) \to \tilde{F}^+_0(K)$ is easily seen to be marked anodyne.

Let $\tilde{i}_0 \colon \tilde{F}_0^+(K) \to F_1^+(K)$ denote the map induced by i_0. We will prove that it is marked anodyne as well. By construction of i_0 as a colimit, it is enough to prove the claim for $K = \Delta^{m-1}$, in which case the morphism $\tilde{i}_0$ is the canonical inclusion $\Delta^m \times \Delta^1 \to \Delta^m * \Delta^m$, where the non-trivial marking occurs in the second copy of Δ^m. Therefore, we are left with showing the following combinatorial result. $\square$

Lemma 4.3.5. *For every $m \in \mathbb{N}$, consider the canonical inclusion $\tilde{i}_0 \colon \Delta^m \times \Delta^1 \to \Delta^m * \Delta^m$, where both simplicial sets are endowed with the minimal marking that makes $(\Delta^m)^\sharp \times \{1\}$ a marked simplicial subset of them. Then $\tilde{i}_0$ is marked anodyne.*

Proof. Note that the inclusion $\tilde{i}_0$ factors through the simplicial set

$$A^m = (\Delta^m \times \Delta^1) \bigcup_{\Delta^{\overline{01}\dots\overline{m}} \underset{\Delta^{\overline{m}}}{\cup} \Delta^{m\overline{m}}} \Delta^{m\overline{01}\dots\overline{m}}. \tag{4.10}$$

First, we want to show that the inclusion $A^m \to \Delta^m * \Delta^m$ is marked anodyne. To this purpose, we consider the inclusion $s_m \colon S_m \to \Delta^m * \Delta^m$, where S_m denotes the spine $\mathrm{Sp}^{0\dots m\overline{0}\dots\overline{m}}$, endowed with the maximal marking that turns s_m into a morphism of marked simplicial sets. As s_m is clearly marked anodyne and factors through A^m, it will suffice to show that the inclusion $S_m \to A^m$ is marked anodyne. The underlying simplicial set of A^m is a union of $m+2$ simplices of dimension $(m+1)$, denoted $\tau_0, \dots, \tau_m, \overline{\tau}$ and defined as follows:

- for $0 \leqslant k \leqslant m$, the simplex τ_k is defined as $\Delta^{01\dots k\,\overline{k}\,\overline{(k+1)}\dots\overline{m}}$,
- the simplex $\overline{\tau}$ is $\Delta^{m\overline{0}\,\overline{1}\dots\overline{m}}$.

In each case, the marking is induced by that of A^m. Writing T_k for $S_m \cup \overline{\tau} \cup \tau_m \cup \dots \cup \tau_k$, we get a filtration of A^m of the form

$$S_m \;\subset\; S_m \cup \overline{\tau} \;\subset\; T_m \;\subset\; T_{m-1} \;\subset\; \dots \;\subset\; T_0 = A^m. \tag{4.11}$$

We will prove that at each step, the inclusion is a marked anodyne morphism. This is clear for $S_m \subset S_m \cup \overline{\tau}$. The second inclusion $S_m \cup \overline{\tau} \subset T_m$ is obtained as the pushout

$$\begin{tikzcd}
(\mathrm{Sp}^{0\dots m\overline{m}})^\flat \arrow[r] \arrow[d] & S_m \cup \overline{\tau} \arrow[d] \\
\tau_m \arrow[r] \arrow[ur, phantom, "\ulcorner"] & T_m
\end{tikzcd} \tag{4.12}$$

and is therefore marked anodyne, since so is the left vertical map. We now prove that the inclusion $T_k \subset T_{k-1}$ is marked anodyne for every $0 < k \leqslant m$. We introduce the marked simplicial set $\tau_k\langle k\rangle$ defined as the face opposed to vertex k in τ_k, or more explicitly as $\Delta^{0\dots(k-1)\overline{k}\dots\overline{m}}$, endowed with the induced marking. The intersection $\tau_k' = T_k \cap \tau_{k-1}$ can then be expressed as

$$\tau_k' = \tau_k\langle k\rangle \bigcup_{(\Delta^{\overline{k}\dots\overline{m}})^\sharp} (\Delta^{\overline{(k-1)}\,\overline{k}\dots\overline{m}})^\sharp \tag{4.13}$$

and the inclusion $T_k \subset T_{k-1}$ as the pushout

$$\begin{tikzcd} \tau'_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T_k \arrow[d] \\ \tau_{k-1} \arrow[r] & T_{k-1}. \end{tikzcd} \tag{4.14}$$

To see that $T_k \to T_{k-1}$ is marked anodyne, it therefore suffices to show that $\tau'_k \to \tau_{k-1}$ has this property. Looking at equation (4.13), we observe that the latter inclusion is of the form $(I_0 \subset I) \circledast (J_0 \subset J)$, with $I_0 = \emptyset$, $I = [k-1]$, $J_0 = \{\overline{k}, \ldots, \overline{m}\}$ and $J = J_0 \cup \{\overline{(k-1)}\}$. By lemma A.1.10, we deduce that this map is marked anodyne, as desired.

It remains to prove that the inclusion $\Delta^m \times \Delta^1 \to A^m$ is marked anodyne. By equation (4.10), it suffices to prove that the inclusion $\Delta^{\overline{0}\ldots\overline{m}} \cup \Delta^{mF\overline{m}} \hookrightarrow \Delta^{m\overline{0}\ldots\overline{m}}$ is marked anodyne. Denoting $B_i = \Delta^{\overline{0}\ldots\overline{m}} \cup \Delta^{m\overline{i}\ldots\overline{m}}$, the latter map can be written as the composite

$$B_m \subset B_{m-1} \subset \ldots \subset B_0.$$

Each inclusion $B_{i+1} \subset B_i$ is induced by its restriction $B_{i+1} \cap \Delta^{m\overline{i}\ldots\overline{m}} \subset B_i \cap \Delta^{m\overline{i}\ldots\overline{m}}$, which is marked anodyne by lemma A.1.9 applied with $I_0 = \emptyset$, $I = \{m\}$, $J = \{\overline{i}, \ldots, \overline{m}\}$ and $J_0 = J \setminus \{\overline{i}\}$; therefore so is the map $\Delta^m \times \Delta^1 \to A^m$. This completes the proof of lemma 4.3.5. □

4.3.3 Study of the maps i_1 and i_2

Lemma 4.3.6. *For K a simplicial set, the morphisms $k_1(K)\colon F_2^+(K) \to F_{2,1,\mathcal{J}}^+(K)$ and $k_2(K)\colon F_2^+(K) \to F_{2,3,\mathcal{J}}^+(K)$ are monomorphisms which are bijective on 0-simplices.*

Proof. The argument is completely analogous to that of the proof of lemma 4.3.3. □

Lemma 4.3.7. *For all simplicial set K, the maps $i_1\colon F_2^+(K) \to F_1^+(K)$ and $i_2\colon F_3^+(K) \to F_2^+(K)$ are marked anodyne.*

Proof. We only give the proof for i_1, the case of the map i_2 being very similar. Recall that i_1 is defined by taking the colimit over all simplices $\Delta^m \to K^{\triangleleft}$ of the morphisms $i_{1,m}\colon (\Delta^m)^{\triangleright} \to F_1^+(K)$, sending Δ^m to the first copy of itself in $(\Delta^m)^{*2} \subseteq F_1^+(K)$ and $\triangleright$ to $\overline{\triangleleft}$. Note that we may restrict the colimit to those simplices Δ^m that contains the cone point $\triangleleft$, in which case the map $i_{1,m}$ factors through $(\Delta^m)^{*2}$. Let $\Delta^m \to K^{\triangleleft}$ be such a simplex; it now suffices to show that the factored map $\overline{i}_{1,m}\colon (\Delta^m)^{\triangleright} \to (\Delta^m)^{*2}$ is marked anodyne.

For the rest of this proof, we relabel $\triangleleft$ as 0, so that we can identify $\overline{i}_{1,m}$ with the obvious inclusion of $(\Delta^m)^{\triangleright} \cong \Delta^{0\ldots m\overline{0}}$ into $(\Delta^m)^{*2} \cong \Delta^{0\ldots m\overline{0}\ldots\overline{m}}$. Now this map is the composite of the sequence

$$\Delta^{0\ldots m\overline{0}} \subset \Delta^{0\ldots m\overline{0}\overline{1}} \subset \ldots \Delta^{0\ldots m\overline{0}\overline{1}\ldots\overline{j}} \subset \ldots \Delta^{0\ldots m\overline{0}\overline{1}\ldots\overline{m}}.$$

Since all edges of the form $\overline{\ell} \to \overline{p}$ are marked, by lemma A.1.10 we deduce that each of these inclusions is marked anodyne, whence the result. □

4.3.4 Study of the map p

This subsection is devoted to the last step of the comparison described in zigzag (4.4), namely we prove the following result.

Proposition 4.3.8. *The functor $p^*\colon \mathcal{C}_3 \to \mathcal{B}\mathcal{O}_\sigma$ induced by the natural transformation $p\colon G^+ \to F_3^+$ is an equivalence.*

We start with some preliminary results.

Lemma 4.3.9. *The functors $F_3, G\colon \mathsf{sSet} \to \mathsf{sSet}$ both preserve monomorphisms.*

Proof. Since s_* and $(-)^\triangleleft$ preserves monomorphisms, so does their composite F_3. We now turn to the case of G. Let $\varphi\colon A \to B$ be a monomorphism of simplicial sets and $n \in \mathbb{N}$. Let a_0 and a_1 be two n-simplices of $G^+(A)$ whose image under $G^+(\varphi)$ coincide. We wish to prove that $a_0 = a_1$. Since $s_*(A \times \Delta^1) \to G^+(A)$ is an epimorphism, we may choose lifts $\widetilde{a_0}$ and $\widetilde{a_1}$ in $(s_*(A \times \Delta^1))_n$ of the two simplices. We distinguish several cases in the argument.

(1) If both $\widetilde{a_0}$ and $\widetilde{a_1}$ belong to the subset $(s_*A)_n$, then their common image $\varphi(a_0) = \varphi(a_1)$ actually belongs to $(s_*\{0\})_n$, which is a subset of $G^+(A)_n$, so that $a_0 = a_1$.

(2) If none of $\widetilde{a_0}$ and $\widetilde{a_1}$ belongs to the subset $(s_*A)_n$, since $s_*(A \times \Delta^1)_n \setminus s_*(A)_n \to G^+(B)_n$ is an inclusion, we deduce that $\widetilde{a_0} = \widetilde{a_1}$, so that again $a_0 = a_1$.

(3) Finally, the case where exactly one of $\widetilde{a_0}$ and $\widetilde{a_1}$ belong to $s_*(A)_n$ is contradictory: indeed, by construction of $G^+(A)$ the subset $s_*(A)_n$ and its complement in $s_*(A \times \Delta^1)_n$ remain disjoint in the quotient $G^+(A)_n$, hence also in $G^+(B)_n$.

□

Lemma 4.3.10. *For every simplicial set K, the morphism $p(K)\colon G^+(K) \to F_3^+(K)$ is an equivalence of marked simplicial sets.*

Proof. Let $\mathcal{U}$ be the class of simplicial sets X for which $p(X)$ is an equivalence. We will show that $\mathcal{U} = \mathsf{sSet}$ by proving that $\mathcal{U}$ contains every representable Δ^m and is stable under isomorphisms, small coproducts, pushouts along monomorphisms and sequential colimits along monomorphisms. In other words, we will show that $\mathcal{U}$ is *saturated by monomorphisms* in the sense of [Cis19, Definition 1.3.9] that contains all representables.

We first show that all standard simplices are in $\mathcal{U}$. Let $m \in \mathbb{N}$ and consider $p(\Delta^m)\colon G^+(\Delta^m) \to F_3^+(\Delta^m)$. This morphism admits a section e, *non-naturally* in the variable $[m] \in \Delta$, whose underlying morphism of simplicial sets is defined as the composition

$$e\colon F_3(\Delta^m) = s_*\Delta^{\triangleleft 0\ldots m} \cong s_*\Delta^{00,01,\ldots,0m} \subset s_*(\Delta^m \times \Delta^1) \to G(\Delta^m).$$

where the middle identification of $(m+1)$-dimensional simplices sends $\triangleleft$ to 00 and the vertex i to $0i$ (following the notations of sections 4.2.2 and 4.2.3). Let D_i be the non-degenerate $(m+1)$-simplex of $\Delta^m \times \Delta^1$ containing the edge $i0 \to i1$, where we declare all edges contained in $\Delta^m \times \{0\}$ to be marked. Let $D_{i,i+1}$ denote the face of D_i opposed to vertex $i1$. The morphism e then factors through a filtration

$$F_3^+(\Delta^m) \xrightarrow{e_0} G_0^+ \xrightarrow{e_1} G_1^+ \longrightarrow \dots \xrightarrow{e_m} G_m^+ = G^+(\Delta^m),$$

where G_i^+ is the image of $s_*(\bigcup_{j=0}^i D_j)$ by the quotient map $s_*(\Delta^m \times \Delta^1) \to G^+(\Delta^m)$, with the induced marking. Since the edge $i0 \to (i+1)0$ is marked, the inclusion $s_*(D_{i,i+1}) \to s_*(D_{i+1})$ is marked anodyne, hence so is its pushout

$$s_* \bigcup_{j=0}^{i} D_j \xrightarrow{\sim} s_* \bigcup_{j=0}^{i+1} D_j.$$

We thus obtain a diagram

$$\begin{tikzcd}
s_*\{0\} \arrow[d, equal] & s_*\Delta^{00,01,\dots,i0} \arrow[l] \arrow[r, tail] \arrow[d, "\wr"] & s_* \bigcup_{j=0}^{i} D_j \arrow[d, "\wr"] \\
s_*\{0\} & s_*\Delta^{00,01,\dots,(i+1)0} \arrow[l] \arrow[r, tail] & s_* \bigcup_{j=0}^{i+1} D_j
\end{tikzcd}$$

where each row defines a cofibrant diagram in the projective model structure on the category of diagrams $\mathrm{Fun}(\bullet \leftarrow \bullet \to \bullet, \mathsf{sSet}^+)$. Taking pushouts yields a weak equivalence $e_i \colon G_i \xrightarrow{\sim} G_{i+1}$. This proves that e is a marked equivalence, hence $\Delta^m \in \mathcal{U}$.

It is clear that $\mathcal{U}$ is stable by isomorphisms. Consider a simplicial set of the form $K = \coprod_{i \in J} K_i$ with all the K_i in $\mathcal{U}$. As $(-)^{\triangleleft} \colon \mathsf{sSet} \to \mathsf{sSet}_*$ and s_* both preserve colimits, we can compute

$$\begin{aligned}
F_3^+(K) &= s_*\left(\left(\coprod_{i\in J} K_i\right)^{\triangleleft}\right) \\
&\cong s_*\left(\left(\coprod_{i\in J} K_i^{\triangleleft}\right) / \coprod_{i\in J} \triangleleft_i\right) \\
&\cong \operatorname{colim}\left(\coprod_{i\in J} s_*(K_i^{\triangleleft}) \longleftarrow \coprod_{i\in J} s_*\{\triangleleft_i\} \longrightarrow s_*\{*\}\right)
\end{aligned}$$

and

$$\begin{aligned}
G^+(K) &\cong \operatorname{colim}\left(\coprod_{i\in J} s_*(K_i \times \Delta^1) \longleftarrow \coprod_{i\in J} s_*(K_i \times \{0\}) \longrightarrow s_*\{0\}\right) \\
&\cong \operatorname{colim}\left(\coprod_{i\in J} G^+(K_i) \longleftarrow \coprod_{i\in J} s_*\{0\} \longrightarrow s_*\{0\}\right).
\end{aligned}$$

Since each map $p(K_i)\colon G^+(K_i) \to F_3^+(K_i) = s_*(K_i^{\triangleleft})$ is a marked equivalence and the diagrams are cofibrant, we deduce that $p(K)$ is also an equivalence. This proves that $\mathcal{U}$ is stable under small colimits.

Suppose now that K is a pushout $B \coprod_A C$, with A, B and C in $\mathcal{U}$ and $A \to B$ a monomorphism. We will prove that $K \in \mathcal{U}$. As before, rewriting the colimits gives isomorphisms

$$F_3^+(K) \cong F_3^+(B) \coprod_{F_3^+(A)} F_3^+(C) \qquad \text{and} \qquad G^+(K) \cong G^+(B) \coprod_{G^+(A)} G^+(C).$$

To prove that $p(K)$ is a weak equivalence, it suffices to show that the pushout diagrams $F_3^+(B) \leftarrow F_3^+(A) \to F_3^+(C)$ and $G^+(B) \leftarrow G^+(A) \to G^+(C)$ are cofibrant. This is a consequence of the fact that F_3^+ and G^+ both preserve monomorphisms; in the first case, this is clear whereas in the latter, it is given by lemma 4.3.9. Finally, the proof that $\mathcal{U}$ is stable by sequential colimits along monomorphisms comes from a similar argument, therefore we omit it. □

Recall from 4.3.1 the notation $\mathcal{J}$ for the standard interval object. We will use the fact that Joyal's model structure on sSet can be obtained à la Cisinski using $\mathcal{J} \times (-)$ as an exact cylinder [Cis19].

Lemma 4.3.11. *For every simplicial set K, the image of the categorical equivalence $q\colon K \times \mathcal{J} \to K$ under the functors F_3^+ and G^+ is a marked equivalence.*

Proof. Let $\mathcal{U}_F$ (respectively $\mathcal{U}_G$) be the class of simplicial sets X such that $F_3^+(q)\colon F_3^+(X \times \mathcal{J}) \to F_3^+(X)$ (resp. $G^+(q)\colon G^+(X \times \mathcal{J}) \to G^+(X)$) is a marked equivalence. We aim at proving that $\mathcal{U}_F = \mathcal{U}_G = \mathsf{sSet}$. Using arguments similar to those of the proof of lemma 4.3.10, one easily shows that both $\mathcal{U}_F$ and $\mathcal{U}_G$ are stable under isomorphisms, small coproducts, pushouts along a monomorphism and sequential colimits along monomorphisms. It is thus enough to prove that $\mathcal{U}_F$ and $\mathcal{U}_G$ contains every representable Δ^m. The key point is the observation that both F_3 and G restricts to an endofunctor on the full subcategory of sSet given by the essential image of the nerve functor from 1-categories. More precisely, we have

$$F_3(\Delta^m \times \mathcal{J}) \cong N\left(s_*([m] \times \mathsf{J})^{\triangleleft}\right)$$

and

$$G(\Delta^m \times \mathcal{J}) \cong N\left(s_*([m] \times \mathsf{J} \times [1]) \coprod_{s_*([m]\times\mathsf{J}\times\{0\})} s_*\{0\}\right),$$

from which one readily verifies that $F_3^+(q)$ and $G^+(q)$ are the image under the nerve functor of an equivalence of 1-categories, hence are marked equivalence. □

We now turn to the proof of the last step in our comparison of $\mathcal{B}\mathcal{O}_\sigma$ and $\mathrm{Ext}(\sigma)$, namely the proof that $p^*\colon \mathcal{C}_3 \to \mathcal{B}\mathcal{O}$ is an equivalence of ∞-categories.

Proof of proposition 4.3.8. Let K be a simplicial set. Recall that $\mathcal{B}\mathcal{O}_\sigma$ is defined by identifying, naturally in K, the set $\mathrm{Hom}(K, \mathcal{B}\mathcal{O}_\sigma)$ with the subset

$\mathrm{Hom}^\sigma(G^+(K), \mathcal{O}^{\otimes,\natural})$ of $\mathrm{Hom}_{\mathsf{sSet}^+}(G^+(K), \mathcal{O}^{\otimes,\natural})$ of morphisms satisfying conditions $(\star)_{G,\sigma}$. Similarly, $\mathrm{Hom}(K, \mathcal{C}_3)$ is identified with the set $\mathrm{Hom}^\sigma(F_3^+(K), \mathcal{O}^{\otimes,\natural})$. Since $p(K)$ preserves the conditions $(\star)_{G,\sigma}$, the map

$$p(K)^*\colon \mathrm{Hom}(F_3^+(K), \mathcal{O}^{\otimes,\natural}) \to \mathrm{Hom}(G^+(K), \mathcal{O}^{\otimes,\natural})$$

restricts to a map $\mathrm{Hom}^\sigma(F_3^+(K), \mathcal{O}^{\otimes,\natural}) \to \mathrm{Hom}^\sigma(G^+(K), \mathcal{O}^{\otimes,\natural})$.

We will need to consider two quotients of these Hom sets, whose associated equivalence relations we will denote $\sim$ and $\approx$. They correspond respectively to the homotopy relations in $\mathrm{Fun}(K, \mathcal{C}_3)$ and $\mathrm{Map}^\flat(F_3^+(K), \mathcal{O}^{\otimes,\natural})$. We only describe those two relations in the case of $\mathrm{Hom}^\sigma(F_3^+(K), \mathcal{O}^{\otimes,\natural})$, the case of $\mathrm{Hom}^\sigma(G^+(K), \mathcal{O}^{\otimes,\natural})$ being similar.

- *(Definition of $\sim$).* First, we consider the set of connected components $\pi_0 \mathrm{Map}(K, \mathcal{C}_3)$, which is defined as the quotient of $\mathrm{Hom}(F_3^+(K), \mathcal{O}^{\otimes,\natural})$ by the homotopy equivalence relation $\sim$ in the functor ∞-category $\mathrm{Fun}(K, \mathcal{C}_3)$.
- *(Definition of $\approx$).* Second, we consider the set of connected components $\pi_0 \mathrm{Map}^\sharp(F_3^+(K), \mathcal{O}^{\otimes,\natural})$, that is the quotient of $\mathrm{Hom}(F_3^+(K), \mathcal{O}^{\otimes,\natural})$ by the homotopy equivalence relation $\approx$ in the functor ∞-category $\mathrm{Map}^\flat(F_3^+(K), \mathcal{O}^{\otimes,\natural})$.

Using the characterization of equivalences in functor ∞-categories of [Lur22, Theorem 01KA] (or more precisely, a slight generalization of this result to marked simplicial sets), we can rephrase the definitions of $\sim$ and $\approx$ more explicitly.

Let f_0 and f_1 be two maps $F_3^+(K) \to \mathcal{O}^{\otimes,\natural}$.

$\boxed{\sim}$ Both of the following conditions are equivalent to asserting that $f_0 \sim f_1$.

($i_\sim$) There exists a factorization of the fold map $(\mathrm{id}_K, \mathrm{id}_K)$ as $K \coprod K \to \overline{K} \xrightarrow{\rho} K$, with ρ a categorical equivalence, and a lift in the diagram

$$\begin{tikzcd} F_3^+(K)^{\amalg 2} \arrow[r, "(f_0,f_1)"] \arrow[d, "(F_3^+(i_0),F_3^+(i_1))"'] & \mathcal{O}^{\otimes,\natural} \\ F_3^+(\overline{K}) \arrow[ur, dashed, bend right, "\overline{f}"'] & \end{tikzcd}$$

such that $\overline{f}$ satisfies conditions $(\star)_{G,\sigma}$.

($ii_\sim$) For every factorization of the fold map $(\mathrm{id}_K, \mathrm{id}_K)$ as $K \coprod K \xrightarrow{(s_0,s_1)} \overline{K} \to K$, where s_0 and s_1 are disjoint monomorphisms, there exists a lift in the diagram

$$\begin{tikzcd} F_3^+(K)^{\amalg 2} \arrow[r, "(f_0,f_1)"] \arrow[d, "(F_3^+(s_0),F_3^+(s_1))"'] & \mathcal{O}^{\otimes,\natural} \\ F_3^+(\overline{K}) \arrow[ur, dashed, bend right, "\overline{f}"'] & \end{tikzcd}$$

such that $\overline{f}$ satisfies conditions $(\star)_{\sigma}$.

$\boxed{\approx}$ Both of the following conditions are equivalent to asserting that $f_0 \approx f_1$.

$(i_\approx)$ There exists a factorization of the fold map $\left(\mathrm{id}_{F_3^+(K)}, \mathrm{id}_{F_3^+(K)}\right)$ as

$$F_3^+(K) \coprod_{\Delta^1} F_3^+(K) \xrightarrow{\iota} \overline{F_3^+(K)} \xrightarrow{\rho} F_3^+(K),$$

with ρ a cartesian equivalence (in the sense of [Lur09a]) , and a lift in the diagram

$$\begin{tikzcd} F_3^+(K) \coprod_{\Delta^1} F_3^+(K) \arrow[r, "{(f_0,f_1)}"] \arrow[d, "\iota"'] & \mathcal{O}^{\otimes,\natural}. \\ \overline{F_3^+(K)} \arrow[ru, dashed, bend right, "\overline{f}"] & \end{tikzcd}$$

$(ii_\approx)$ For every factorization of the fold map $\left(\mathrm{id}_{F_3^+(K)}, \mathrm{id}_{F_3^+(K)}\right)$ as

$$F_3^+(K) \coprod_{\Delta^1} F_3^+(K) \xrightarrow{\iota} \overline{F_3^+(K)} \xrightarrow{\rho} F_3^+(K),$$

where ι is a monomorphism, there exists a lift in the diagram

$$\begin{tikzcd} F_3^+(K) \coprod_{\Delta^1} F_3^+(K) \arrow[r, "{(f_0,f_1)}"] \arrow[d, "\iota"'] & \mathcal{O}^{\otimes,\natural}. \\ \overline{F_3^+(K)} \arrow[ru, dashed, bend right, "\overline{f}"] & \end{tikzcd}$$

Using these descriptions, one easily sees that $p(K)^*$ induces maps on the quotients by the equivalence relations $\sim$ and $\approx$, that we denote respectively

$$\begin{aligned} p^*_{\sigma,\sim} &: \mathrm{Hom}^\sigma(F_3^+(K), \mathcal{O}^{\otimes,\natural})_{/\sim} \longrightarrow \mathrm{Hom}^\sigma(G^+(K), \mathcal{O}^{\otimes,\natural})_{/\sim}, \\ p^*_{\sigma,\approx} &: \mathrm{Hom}^\sigma(F_3^+(K), \mathcal{O}^{\otimes,\natural})_{/\approx} \longrightarrow \mathrm{Hom}^\sigma(G^+(K), \mathcal{O}^{\otimes,\natural})_{/\approx}. \end{aligned}$$

By lemma 4.3.10, we know that $p(K)$ is marked weak equivalence, so that

$$p^*_\approx \colon \mathrm{Hom}(F_3^+(K), \mathcal{O}^{\otimes,\natural})_{/\approx} \longrightarrow \mathrm{Hom}(G^+(K), \mathcal{O}^{\otimes,\natural})_{/\approx}$$

is a bijection. It is an easy observation that a morphism $f\colon F_3^+(K) \to \mathcal{O}^{\otimes,\natural}$ verifies the conditions $(\star)_{3,\sigma}$ if and only if $f \circ p(K)\colon G^+(K) \to \mathcal{O}^{\otimes,\natural}$ satisfies the corresponding condition $(\star)_{G,\sigma}$. Therefore the induced map $p^*_{\sigma,\approx}$ is a bijection. The fact that $p^*_{\sigma,\sim}$ is a bijection is now a consequence of the following result.

Claim. The equivalence relations $\sim$ and $\approx$ coincide, both on $\mathrm{Hom}(F_3^+(K), \mathcal{O}^{\otimes,\natural})$ and on $\mathrm{Hom}(G^+(K), \mathcal{O}^{\otimes,\natural})$.

We prove the claim for the functor F_3^+, the case of G^+ being similar. Consider two morphisms $f_0, f_1\colon F_3^+(K) \to \mathcal{O}^{\otimes,\natural}$.

$\boxed{\sim\Rightarrow\approx}$ Suppose $f_0 \sim f_1$. By assumption $(ii_\sim)$, there exists a lift $\overline{f}$ in the diagram

$$\begin{tikzcd} F_3^+(K)^{\amalg 2} \arrow[r, "(f_0,f_1)"] \arrow[d, "(F_3^+(s_0),F_3^+(s_1))"'] & \mathcal{O}^{\otimes,\natural} \\ F_3^+(K\times\mathcal{J}) \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

and by lemma 4.3.11, the map $F_3^+(K\times\mathcal{J}) \to F_3^+(K)$ is a marked equivalence, so that condition $(i_\approx)$ is satisfied.

$\boxed{\sim\Leftarrow\approx}$ Suppose that $f_0 \approx f_1$. We will show that condition $(i_\sim)$ holds. The factorization of $(\mathrm{id}_K, \mathrm{id}_K)$ through $q\colon K\times\mathcal{J}\to K$ induces a factorization

$$\left(\mathrm{id}_{F_3^+(K)}, \mathrm{id}_{F_3^+(K)}\right) : F_3^+(K)\coprod_{\Delta^1} F_3^+(K) \longrightarrow F_3^+(K\times\mathcal{J}) \longrightarrow F_3^+(K)$$

where the first map is a monomorphism. We can thus apply assumption $(ii_\approx)$ to obtain a lift $\overline{f}$ in the diagram

$$\begin{tikzcd} F_3^+(K)^{\amalg 2} \arrow[r, "(f_0,f_1)"] \arrow[d, "(F_3^+(s_0),F_3^+(s_1))"'] & \mathcal{O}^{\otimes,\natural}. \\ F_3^+(K\times\mathcal{J}) \arrow[ur, dashed, "\overline{f}"'] & \end{tikzcd}$$

It now suffices to prove that $\overline{f}$ satisfies conditions $(\star)_{3,\sigma}$, which is a consequence of the fact that f_0 and f_1 both do and $(K\times\mathcal{J})_0 \cong (K^{\coprod 2})_0$.

This shows the above claim and therefore completes our proof of proposition 4.3.8.

□

Proof of theorem 4.1.1. Combining lemma 4.3.2, proposition 4.3.4, lemma 4.3.7 and proposition 4.3.8, we obtain that each map in the zigzag (4.4) is an equivalence, so that $\mathrm{Ext}(\sigma)$ and $\mathcal{B}\mathcal{O}_\sigma$ are equivalent Kan complexes. □

Chapter 5

Homotopy type of spaces of extensions

Contents

5.1 Statement of the results

5.1.1 Motivation

Let $\mathcal{O}^{\otimes}$ be a unital ∞-operad, that we now assume to be monochromatic and such that the underlying ∞-category is an ∞-groupoid. Consider an operation $\sigma \in \mathcal{O}(n)$ of arity n.

In the previous chapter, we provided a zigzag of equivalences between two models for the space of extensions of σ: on the one hand, the fiber $\mathcal{BO}_\sigma$ of Mann–Robalo's brane fibration $\pi\colon \mathcal{BO} \to \mathrm{Tw}(\mathrm{Env}(\mathcal{O}))^{\otimes}$ and on the other hand, Lurie's space $\mathrm{Ext}(\sigma)$. As explained in the introduction under the name of problem C, neither of these two models is suitable for applications, since computing their homotopy type seems difficult *even for simple examples of* ∞*-operads.*

However, there is a third possible model for the space of extensions of the operation σ, namely Toën's space $\mathcal{E}\mathrm{xt}_\sigma$, defined as the homotopy fiber

$$\mathcal{O}(n+1) \underset{\mathcal{O}(n)}{\overset{\mathrm{h}}{\times}} \{\sigma\},$$

of the morphism $i^*\colon \mathcal{O}(n+1) \to \mathcal{O}(n)$ given by precomposition with a *chosen atomic map* i.

In this chapter, we compare Toën's model $\mathcal{E}\mathrm{xt}_\sigma$ to Lurie's space $\mathrm{Ext}(\sigma)$. We will show that, contrary to what one might expect, the models are not equivalent in general, unless the space $\mathcal{O}(1)$ of unary operations is contractible. More precisely, we will prove the following result.

Theorem 5.1.1 (Theorem C)**.** *Let $\mathcal{O}^\otimes$ be a monochromatic unital ∞-operad whose underlying ∞-category $\mathcal{O}$ is an ∞-groupoid and $\sigma \in \mathcal{O}(n)$ an operation of arity n. Choose an atomic morphism $i\colon \langle n\rangle \to \langle n+1\rangle$ in $\mathcal{O}^\otimes$. Then there is a homotopy cartesian square*

$$\begin{tikzcd} \mathcal{O}(n+1) \underset{\mathcal{O}(n)}{\overset{\mathrm{h}}{\times}} \{\sigma\} \arrow[r] \arrow[d] & \mathrm{Ext}(\sigma) \arrow[d] \\ * \arrow[r] & B\mathcal{O}(1), \end{tikzcd} \tag{5.1}$$

well-defined in the homotopy category of spaces, which exhibits $\mathrm{Ext}(\sigma)$ *as a homotopy quotient of* $\mathcal{O}(n+1) \underset{\mathcal{O}(n)}{\overset{\mathrm{h}}{\times}} \{\sigma\}$ *by a* $\mathcal{O}(1)$*-action.*

Remark 5.1.2. The restriction to the monochromatic situation is merely there to make the comparison with Toën's model more transparent and to slightly simplify the notations. The results of this chapter readily generalize to the general case of (coloured) unital ∞-operads.

Notation 5.1.3. Throughout this chapter, when considering a monochromatic ∞-operad $\mathcal{O}^\otimes$ with unique color c, we shall use the slightly abusive notation of writing objects of $\mathcal{O}^\otimes$ in the form $\langle n\rangle$, instead of say $c^{\oplus n}$.

5.1.2 Difference with the existing literature

Theorem 5.1.1 contradicts the statement [Lur17, Remark 5.1.1.10], in the case of ∞-operads with non-contractible spaces of unary operations. This statement is a key result in Lurie's proof of coherence of the little disks ∞-operad $\mathbb{E}_n$. Note that this statement is, however, *only used* for this example of $\mathbb{E}_n$, which satisfies that $\mathbb{E}_n(1) \simeq *$ so that our theorem 5.1.1 actually implies that the conclusion of Lurie's statement is true, *in this particular case.*

We then provide through proposition 5.1.4 an explicit example of an ∞-operad for which this statement is incorrect, without appealing to the above theorem 5.1.1.

Let us now explain Lurie's statement. Let $\mathcal{O}_\Delta$ be a unital fibrant simplicial operad, with underlying ∞-operad $\mathcal{O}^\otimes = \mathcal{N}^\otimes(\mathcal{O}_\Delta)$. By unitality, the canonical inclusion $i_{\mathbb{F}_*} \colon \langle m\rangle \to \langle m+1\rangle$ in $\mathbb{F}_*$ lifts uniquely to a morphism i in the simplicial category $\mathcal{O}^\otimes_\Delta$, which induces a map of simplicial sets

$$i^* \colon \operatorname{Map}^{\mathrm{act}}_{\mathcal{O}^\otimes_\Delta}(\langle m+1\rangle, \langle n\rangle) \to \operatorname{Map}^{\mathrm{act}}_{\mathcal{O}^\otimes_\Delta}(\langle m\rangle, \langle n\rangle).$$

Given an active morphism $f \colon \langle m\rangle \to \langle n\rangle$ in $\mathcal{O}^\otimes_\Delta$, Lurie defines in [Lur17, Notation 5.1.1.8.] the *space of strict extensions of* f, denoted $\operatorname{Ext}_\Delta(f)$, as the fiber of i^* at f. Now consider an n-simplex σ of $\mathcal{O}^\otimes$ corresponding to a sequence of n composable active morphisms

$$\langle m_0\rangle \xrightarrow{f_1} \langle m_1\rangle \xrightarrow{f_2} \dots \xrightarrow{f_n} \langle m_n\rangle$$

in $\mathcal{O}^\otimes_\Delta$. In [Lur17, Construction 5.1.1.9.], a comparison map $\theta \colon \operatorname{Ext}_\Delta(f_n) \to \operatorname{Ext}(\sigma)$ is defined. Then [Lur17, Remark 5.1.1.10] asserts that θ can be identified with the canonical map $\operatorname{fib}_{f_n}(i^*) \to \operatorname{hofib}_{f_n}(i^*)$. In particular, $\operatorname{Ext}(\sigma)$ is supposed to be equivalent to the fiber of i^* at f_n. However, as a consequence of theorem 5.1.1, this equivalence can only hold when the group of unary operations $\mathcal{O}(1)$ is trivial.

For a direct counterexample to [Lur17, Remark 5.1.1.10] for $\mathcal{O}(1) \not\simeq *$, consider the operad $\mathcal{O}_\Delta = \mathrm{AssInv}$ encoding associative algebras together with an involution. It is the monochromatic operad in sets freely generated by two operations $\mu \in \mathcal{O}_\Delta(2)$ and $\tau \in \mathcal{O}_\Delta(1)$ satisfying the relations $\tau^2 = \mathrm{id}$ and $\tau \circ \mu(a,b) = \mu(\tau b, \tau a)$; it can also be described as a semi-direct product $\mathrm{Ass} \rtimes \Sigma_2$. Its homotopy coherent nerve $\mathcal{O}^\otimes = \mathcal{N}(\mathcal{O}^\otimes_\Delta)$ is a unital monochromatic discrete ∞-operad, in which morphisms from $\langle m\rangle$ to $\langle n\rangle$ are given by a map $\alpha \colon \langle m\rangle \to \langle n\rangle$ in $\mathbb{F}_*$, a linear order on each preimage $\alpha^{-1}\{i\}$ and a choice of sign $\varepsilon \colon \langle m\rangle \to \{+,-\}$. The previous morphism will be denoted more compactly

$$\left(k_{i1}^{\varepsilon(k_{i1})} \dots k_{im_i}^{\varepsilon(k_{im_i})}\right)_{i\in\langle n\rangle^\circ}, \text{ with } \alpha^{-1}\{i\} = \{k_{i1} < \dots < k_{im_i}\}.$$

Composition is defined so that negative signs reverse the linear order.

Proposition 5.1.4. *For $\mathcal{O}^\otimes = \mathcal{N}(\mathrm{AssInv})^\otimes$ the ∞-operad of associative algebras with involution and σ the identity operation on the unique color $\langle 1\rangle$, the spaces of extensions* $\operatorname{Ext}(\sigma)$ *and that of strict extensions* $\operatorname{Ext}_\Delta(\sigma)$ *are* not *homotopy equivalent.*

Proof. On the one hand, since $\mathcal{O}_\Delta$ is a discrete simplicial operad, the homotopy fiber $\mathcal{O}_\Delta(2) \underset{\mathcal{O}_\Delta(1)}{\overset{\mathrm{h}}{\times}} \{\sigma\}$ coincide with the actual fiber $\operatorname{Ext}_\Delta(\sigma)$, which is the 4-elements set $\{\mu(a,b), \mu(a,\tau b), \mu(b,a), \mu(\tau b, a)\}$.

On the other hand, we claim that the set of connected components $\pi_0 \operatorname{Ext}(\sigma)$ consists of only two elements. To prove this, recall the description of k-simplices of $\mathrm{Ext}(\sigma)$ given in (2.1). In particular, the objects of $\mathrm{Ext}(\sigma)$ are given by commutative squares

$$\begin{tikzcd} \langle 1\rangle \arrow[r, "\sigma=\mathrm{id}"] \arrow[d, hook, "i"'] & \langle 1\rangle \arrow[d, "\varphi", "\sim"'] \\ \langle 2\rangle \arrow[r, "\alpha"] & \langle 1\rangle. \end{tikzcd} \tag{5.2}$$

with i atomic, α active and φ an equivalence. The morphisms in $\mathrm{Ext}(\sigma)$ are given by diagrams

$$\begin{tikzcd} \langle 1\rangle \arrow[r, "\sigma=\mathrm{id}"] \arrow[d, hook, "i"'] \arrow[dd, bend right=60, "i'"'] & \langle 1\rangle \arrow[d, "\varphi", "\sim"'] \arrow[dd, bend left=60, "\varphi'", "\sim"'] \\ \langle 2\rangle \arrow[r, "\alpha"] \arrow[d, "f"] & \langle 1\rangle \arrow[d, "\psi", "\sim"'] \\ \langle 2\rangle \arrow[r, "\alpha'"] & \langle 1\rangle \end{tikzcd} \tag{5.3}$$

with active morphisms and with f compatible with extensions. Consider two objects $x = (i, \varphi, \alpha)$ and $x' = (i', \varphi', \alpha')$. There exists a unique morphism ψ such that $\psi\varphi = \varphi'$, whereas there are always two distinct morphisms f that are compatible with extensions and satisfy $fi = i'$. For this data (f, ψ) to define a morphism in $\mathrm{Ext}(\sigma)$, we further need equation $\alpha' f = \psi\alpha$ to be satisfied. But since there are 4 active morphisms $\langle 2\rangle \to \langle 1\rangle$ (namely 1^+2^+, 1^+2^-, 2^+1^+ and 2^+1^-), only half of the pairs (x, x') are in the same connected component; this concludes the computation.

□

5.2 Auxiliary models for $\mathrm{Ext}(\sigma)$ and the homotopy fiber of $\mathcal{O}(n+1) \to \mathcal{O}(n)$

The first step is to represent the map $i^*\colon \mathcal{O}(n+1) \to \mathcal{O}(n)$, which is only well-defined in the homotopy category of spaces, by a zigzag of spaces

$$\mathcal{O}(n+1) \xleftarrow{\sim} \widetilde{\mathcal{O}(n+1)} \longrightarrow \mathcal{O}(n).$$

This will give a strict model of the homotopy fiber $\mathcal{O}(n+1) \overset{\mathrm{h}}{\underset{\mathcal{O}(n)}{\times}} \{\sigma\}$.

Consider the subsimplicial set

$$\Delta^2_2 = \Delta^{01} \coprod \Delta^{\{2\}}$$

of Δ^2. Using that Λ^2_1 can be written as the pushout $\Lambda^2_1 = \Delta^2_2 \coprod_{\partial\Delta^{12}} \Delta^{12}$, we obtain that the mapping space $\mathcal{O}(n+1)$ is isomorphic to the fiber

$$\begin{aligned} \widetilde{\mathcal{O}(n+1)} &:= \mathrm{Fun}(\Delta^{12}, \mathcal{O}^{\otimes}_{\mathrm{act}}) \underset{\mathrm{Fun}(\partial\Delta^{12}, \mathcal{O}^{\otimes}_{\mathrm{act}})}{\times} \{(\langle n+1\rangle, \langle 1\rangle)\} \\ &\cong \mathrm{Fun}(\Lambda^2_1, \mathcal{O}^{\otimes}_{\mathrm{act}}) \underset{\mathrm{Fun}(\Delta^2_2, \mathcal{O}^{\otimes}_{\mathrm{act}})}{\times} \{(i, \langle 1\rangle)\}\,. \end{aligned}$$

Similarly, writing Λ^2_0 as $\Delta^2_2 \coprod_{\partial\Delta^{02}} \Delta^{02}$, we can identify $\mathcal{O}(n)$ as the fiber

$$\mathcal{O}(n) \cong \mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}}) \underset{\mathrm{Fun}(\Delta^2_2, \mathcal{O}^{\otimes}_{\mathrm{act}})}{\times} \{(i, \langle 1\rangle)\}\,.$$

Now let $\widetilde{\mathcal{O}(n+1)}$ denote the fiber of the restriction

$$\mathrm{Fun}(\Delta^2, \mathcal{O}^{\otimes}_{\mathrm{act}}) \longrightarrow \mathrm{Fun}(\Delta^2_2, \mathcal{O}^{\otimes}_{\mathrm{act}})$$

at $(i, \langle 1\rangle)$. Since $\Delta^2_2 \hookrightarrow \Delta^2$ factors through both Λ^2_0 and Λ^2_2, we obtain a commutative diagram

$$\begin{tikzcd}
\mathcal{O}(n+1) \arrow[d, "{(i,\mathrm{id})}"] & \widetilde{\mathcal{O}(n+1)} \arrow[l, two heads, "\sim"'] \arrow[r, two heads] \arrow[d] \arrow[dl, phantom, "\llcorner", very near start] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{O}(n) \arrow[d, "{(i,\mathrm{id})}"] \\
\mathrm{Fun}(\Lambda^2_1, \mathcal{O}^{\otimes}_{\mathrm{act}}) & \mathrm{Fun}(\Delta^2, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[l, two heads, "\sim"'] \arrow[r, two heads] & \mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}}).
\end{tikzcd} \tag{5.4}$$

Lemma 5.2.1. *The above diagram yields equivalences of fibers*

$$\mathcal{O}(n+1) \overset{\mathrm{h}}{\underset{\mathcal{O}(n)}{\times}} \{\sigma\} \simeq \widetilde{\mathcal{O}(n+1)} \underset{\mathcal{O}(n)}{\times} \{\sigma\} \cong \mathrm{Fun}(\Delta^2, \mathcal{O}^{\otimes}_{\mathrm{act}}) \underset{\mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}})}{\times} \{(i,\sigma)\}.$$

Proof. In diagram (5.4), observe that the top row is obtained as the fiber of the bottom row at the point $(i, \langle 1\rangle)$ of $\mathrm{Fun}(\Delta^2_2, \mathcal{O}^{\otimes}_{\mathrm{act}})$. Therefore both squares in the diagram are cartesian. In particular, since the bottom right morphism is a Kan fibration, so is $\widetilde{\mathcal{O}(n+1)} \to \mathcal{O}(n)$. Similarly, since the bottom left morphism is a trivial Kan fibration, so is $\widetilde{\mathcal{O}(n+1)} \to \mathcal{O}(n+1)$. This gives the first homotopy equivalence of the lemma. The second equivalence, which is an isomorphism of simplicial sets, is obtained by taking the fiber of the right cartesian square at the object $\sigma \in \mathcal{O}(n)$, whose image in $\mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}})$ is (i,σ). $\square$

We now replace $\mathrm{Ext}(\sigma)$ with a more convenient model, that we shall denote $\mathrm{Ext}^{\square}(\sigma)$, defined as a certain subcategory of the functor ∞-category $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})$ of commutative squares of active maps in $\mathcal{O}^{\otimes}$. The squares will be indexed as follows:

$$\begin{tikzcd}
00 \arrow[r] \arrow[d] & 01 \arrow[d] \\
10 \arrow[r] & 11.
\end{tikzcd}$$

We will require that the left vertical map is atomic, the right vertical one is an equivalence and the top one is precisely the fixed morphism σ. In order to define $\mathrm{Ext}^{\square}(\sigma)$, the following notations will be convenient.

Notation 5.2.2. Let $\mathrm{Atom}_{\mathcal{O}}$ denote the non-full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})$ whose objects are atomic morphisms (see definition 2.2.1). Let $\mathrm{Atom}_{\mathcal{O}}(n)$ be the full subcategory of $\mathrm{Atom}_{\mathcal{O}}$ whose objects are maps with codomain $\langle n\rangle$.

Notation 5.2.3. The marked simplicial set obtained from the square $\Delta^1 \times \Delta^1$ by further marking the edge $\Delta^1 \times \{1\}$ will be denoted $\square$.

Definition 5.2.4 (Definition of $\mathrm{Ext}^{\square}(\sigma)$)**.** Define the simplicial set $\mathrm{Ext}^{\square}(\sigma)$ as the iterated fiber product

$$\mathrm{Ext}^{\square}(\sigma) = \lim \left(\begin{tikzcd} \mathrm{Atom}_{\mathcal{O}} \arrow[d, hook] & & \mathrm{Map}^{\flat}(\square, (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\natural}) \arrow[dll] \arrow[drr] & & \{\sigma\} \arrow[d] \\ \mathrm{Fun}(\Delta^1 \times \{0\}, \mathcal{O}^{\otimes}_{\mathrm{act}}) & & & & \mathrm{Fun}(\{0\} \times \Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}}) \end{tikzcd} \right)$$

where the two diagonal morphisms are the obvious restriction maps.

In other words, $\mathrm{Ext}^{\square}(\sigma)$ is the subcategory of $\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})$ whose

- **objects** are commutative diagrams

$$\begin{tikzcd} \langle n \rangle \arrow[r, "\sigma"] \arrow[d, hook, "\mathrm{atom}"'] & \langle 1 \rangle \arrow[d, "\wr"] \\ \langle n+1 \rangle \arrow[r] & \langle 1 \rangle \end{tikzcd}$$

  in which the left vertical map is atomic and the right vertical map is an equivalence,

- **morphisms** are compatible with extensions, i.e. preserve the new color $\langle n+1 \rangle \setminus \mathrm{Im}(\langle n \rangle)$.

It is easy to see that $\mathrm{Ext}^{\square}(\sigma)$ is an ∞-category.

Lemma 5.2.5. *The space* $\mathrm{Ext}(\sigma)$ *is equivalent to* $\mathrm{Ext}^{\square}(\sigma)$*.*

Proof. By inspection of definition 2.2.3, one easily verifies that all diagrams involved in the definition contains only active maps. Therefore diagrams to $\mathrm{Ext}(\sigma)$ factor through the subcategory $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})_{\sigma/}$ of $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes})_{\sigma/}$. Now recall the canonical equivalence of ∞-categories

$$\gamma\colon \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})_{\sigma/} \xrightarrow{\sim} \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})^{\sigma/}$$

from the slice to the alternative slice, the latter being defined as the fiber at σ of the restriction map

$$\mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}}) \longrightarrow \mathrm{Fun}(\{0\} \times \Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}}).$$

The restriction of γ to $\mathrm{Ext}(\sigma)$ factors through the obvious inclusion $\mathrm{Ext}^{\square}(\sigma) \to \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})^{\sigma/}$. Moreover, by inspection of the objects of these two ∞-categories, one sees that this functor $\gamma|_{\mathrm{Ext}(\sigma)}\colon \mathrm{Ext}(\sigma) \to \mathrm{Ext}^{\square}(\sigma)$ is essentially surjective. To prove the lemma, it therefore suffices to show fully faithfullness of $\gamma|_{\mathrm{Ext}(\sigma)}$. Given two extensions $X, X' \in \mathrm{Ext}(\sigma)$, consider the commutative diagram

$$\begin{tikzcd} \mathrm{Map}_{\mathrm{Ext}(\sigma)}(X, X') \arrow[r] \arrow[d, "\gamma|_{\mathrm{Ext}(\sigma)}"'] & \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})_{\sigma/}}(X, X') \arrow[d, "\wr"', "\gamma"] \\ \mathrm{Map}_{\mathrm{Ext}^{\square}(\sigma)}(\gamma(X), \gamma(X')) \arrow[r] & \mathrm{Map}_{\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})^{\sigma/}}(\gamma(X), \gamma(X')). \end{tikzcd}$$

As γ is an equivalence of ∞-categories, the right vertical map is a homotopy equivalence. Observe that, given two equivalent morphisms $f_0 \simeq f_1 \colon X \to X'$ in $\mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}})_{\sigma/}$, f_0 is compatible with extensions if and only if f_1 has this property, and similarly for morphisms $\gamma(X) \to \gamma(X')$. Consequently, the horizontal maps in the above diagram are both inclusions of the connected components determined by the condition of preservation of the new color in the extensions. Therefore the restriction $\gamma|_{\mathrm{Ext}(\sigma)}$ is a homotopy equivalence, as desired. $\square$

To compare $\mathrm{Ext}^{\square}(\sigma)$ with $\mathcal{O}(n+1) \overset{\mathrm{h}}{\underset{\mathcal{O}(n)}{\times}} \{\sigma\}$, we first give an alternative description of the former ∞-groupoid. By definition of $\mathrm{Atom}_{\mathcal{O}}(n)$, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Atom}_{\mathcal{O}}(n) \arrow[rrr] \arrow[dd] \arrow[drr, hook, dotted, bend right, "j"] \arrow[drr, phantom, "\lrcorner", very near start] & & & \{\sigma\} \arrow[d] \\ & & \mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\{0\} \times \Delta^1, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[d] \\ \mathrm{Atom}_{\mathcal{O}} \arrow[rr] & & \mathrm{Fun}(\Delta^1 \times \{0\}, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[r] & \mathrm{Fun}(\{0\} \times \{0\}, \mathcal{O}^{\otimes}_{\mathrm{act}}) \end{tikzcd} \tag{5.5}$$

in which both squares are cartesian, using the identification

$$\Lambda^2_0 = \Delta^{01} \cup \Delta^{02} \cong \Delta^1 \times \{0\} \cup \{0\} \times \Delta^1, \tag{5.6}$$

and the map j is induced by the universal property of pullbacks.

Lemma 5.2.6. *There is a canonical isomorphism*

$$\mathrm{Ext}^{\square}(\sigma) \cong \mathrm{Map}^{\flat}(\square, (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\natural}) \underset{\mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}})}{\times} \mathrm{Atom}_{\mathcal{O}}(n). \tag{5.7}$$

Proof. Consider the following commutative diagram, extending diagram (5.5):

$$
\begin{tikzcd}
\mathrm{Ext}^{\square}(\sigma) \arrow[rr] \arrow[dr] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd] & & \mathrm{Map}^{\flat}(\square, \mathcal{O}^{\otimes}_{\mathrm{act}}) \underset{(\mathcal{O}^{\otimes}_{\mathrm{act}})^{\{0\}\times\Delta^1}}{\times} \{\sigma\} \arrow[dr] \arrow[drr] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[dd] & & \\
& \mathrm{Atom}_{\mathcal{O}}(n) \arrow[rr] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & & (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\Delta^1\times\{0\}} \underset{\mathcal{O}^{\otimes}_{\mathrm{act}}}{\times} \{\sigma\} \arrow[r] \arrow[dd] \arrow[dr, phantom, "\lrcorner", very near start] & \{\sigma\} \arrow[dd] \\
\mathrm{Atom}_{\mathcal{O}} \underset{(\mathcal{O}^{\otimes}_{\mathrm{act}})^{\Delta^1\times\{0\}}}{\times} \mathrm{Map}^{\flat}(\square, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[rr] \arrow[dr] \arrow[dr, phantom, "\lrcorner", very near start] \arrow[ddr] & & \mathrm{Map}^{\flat}(\square, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[dr] \arrow[drr] \arrow[ddr] & & \\
& \mathrm{Atom}_{\mathcal{O}} \underset{\mathcal{O}^{\otimes}_{\mathrm{act}}}{\times} (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\{0\}\times\Delta^1} \arrow[rr] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & & (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\Lambda^2_0} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\{0\}\times\Delta^1} \arrow[d] \\
& \mathrm{Atom}_{\mathcal{O}} \arrow[rr] & & (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\Delta^1\times\{0\}} \arrow[r] & (\mathcal{O}^{\otimes}_{\mathrm{act}})^{\{0\}\times\{0\}}.
\end{tikzcd}
$$

In the above, certain squares are cartesian by construction, namely:

- all the squares whose arrows are either vertical or horizontal
- the two squares that contains $\mathrm{Map}^{\flat}(\square, \mathcal{O}^{\otimes}_{\mathrm{act}})$ and either $\{\sigma\}$ or $\mathrm{Atom}_{\mathcal{O}}$.

Using the usual transitivity rule for pullback squares, one deduce that any square in the top left cube is cartesian, from which the desired isomorphism follows. $\square$

5.3 Proof of theorem 5.1.1

There are two differences between the homotopy fiber of $\mathcal{O}(n+1) \to \mathcal{O}(n)$ at σ, modelled as $\mathrm{fib}_{(i,\sigma)}(\mathrm{Fun}(\Delta^2, \mathcal{O}^{\otimes}_{\mathrm{act}}) \to \mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}})$, and $\mathrm{Ext}^{\square}(\sigma)$:

(1) objects of $\mathrm{hofib}_{\sigma}(\mathcal{O}(n+1) \to \mathcal{O}(n))$ are given by commutative *triangles* in $\mathcal{O}^{\otimes}_{\mathrm{act}}$, whereas objects of $\mathrm{Ext}^{\square}(\sigma)$ are commutative *squares*,

(2) in $\mathrm{hofib}_{\sigma}(\mathcal{O}(n+1) \to \mathcal{O}(n))$, the map $\langle n \rangle \to \langle n+1 \rangle$ is the fixed morphism i whereas in $\mathrm{Ext}^{\square}(\sigma)$, any atomic map is allowed.

As we will see, the first difference does not affect the homotopy type of the spaces, but the second difference explains the origin of the quotient by the action of $\mathcal{O}(1)$. To make this remark precise, we will introduce variants of $\mathrm{Ext}^{\square}(\sigma)$ that differ according to the previous two parameters.

Consider the morphism $r\colon \Delta^1 \times \Delta^1 \to \Delta^2$ induced from the map of posets $[1] \times [1] \to [2]$ given by

$$r(0,0) = 0, \quad r(0,1) = 1, \quad r(1,0) = 1, \quad r(1,1) = 2.$$

It induces a morphism of marked simplicial sets $\square \to (\Delta^2)^{\flat}$, which yields a restriction map $r^*\colon \mathrm{Fun}(\Delta^2, \mathcal{O}^{\otimes}_{\mathrm{act}}) \to \mathrm{Map}^{\flat}(\square, \mathcal{O}^{\otimes}_{\mathrm{act}})$. Moreover, r extends the identification (5.6) to a commutative square

$$\begin{tikzcd}
\{0\} \times \Delta^1 \cup \Delta^1 \times \{0\} \arrow[r, "\cong"] \arrow[d, hook] & \Lambda^2_0 \arrow[d, hook] \\
\Delta^1 \times \Delta^1 \arrow[r, "r"] & \Delta^2.
\end{tikzcd}$$

Definition 5.3.1. Let $\mathrm{Ext}^{\triangle}(\sigma)$ and $\mathrm{Ext}^{\square}(\sigma, i)$ be the ∞-categories fitting in the following diagram of cartesian squares

$$\begin{tikzcd}
\mathcal{O}(n+1) \underset{\mathcal{O}(n)}{\overset{\mathrm{h}}{\times}} \{\sigma\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ext}^{\triangle}(\sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Fun}(\Delta^2, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[d, "r^*"] \\
\mathrm{Ext}^{\square}(\sigma, i) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ext}^{\square}(\sigma) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Map}^{\flat}(\square, \mathcal{O}^{\otimes}_{\mathrm{act}}) \arrow[d] \\
* \arrow[r, "i"] \arrow[rr, bend right, "\sigma"'] & \mathrm{Atom}_{\mathcal{O}}(n) \arrow[r, "j"] & \mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}}).
\end{tikzcd} \tag{5.8}$$

The fact that $\mathcal{O}(n+1) \underset{\mathcal{O}(n)}{\overset{\mathrm{h}}{\times}} \{\sigma\}$ and $\mathrm{Ext}^{\square}(\sigma)$ fit in the above diagram is a reformulation of lemmas 5.2.1 and 5.2.6.

Lemma 5.3.2. *In the top left square of diagram* (5.8)

$$\begin{tikzcd}
\mathcal{O}(n+1) \underset{\mathcal{O}(n)}{\overset{\mathrm{h}}{\times}} \{\sigma\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ext}^{\triangle}(\sigma) \arrow[d] \\
\mathrm{Ext}^{\square}(\sigma, i) \arrow[r] & \mathrm{Ext}^{\square}(\sigma),
\end{tikzcd} \tag{5.9}$$

the vertical maps are equivalences.

Proof. A simple computation shows that r is an equivalence of marked simplicial sets $\square = (\Delta^1 \times \Delta^1, \Delta^1 \times \{1\}) \to (\Delta^2)^{\flat}$. Therefore r^* is an equivalence of ∞-categories. Since $\mathrm{Map}^{\flat}(\square, \mathcal{O}^{\otimes}_{\mathrm{act}})$ and $\mathrm{Fun}(\Delta^2, \mathcal{O}^{\otimes}_{\mathrm{act}})$ are fibrant over $\mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}})$, taking pullback along the morphisms j and $\sigma\colon * \to \mathrm{Fun}(\Lambda^2_0, \mathcal{O}^{\otimes}_{\mathrm{act}})$ gives the desired equivalences. $\square$

Lemma 5.3.3. *Let $\mathcal{O}^{\otimes}$ be a monochromatic unital ∞-operad. Then the ∞-category $\mathrm{Atom}_{\mathcal{O}}(n)$ is equivalent to the underlying ∞-category $\mathcal{O}$ of $\mathcal{O}^{\otimes}$.*

Corollary 5.3.4. *Let $\mathcal{O}^{\otimes}$ be as above and assume moreover that its underlying ∞-category $\mathcal{O}$ is an ∞-groupoid. Then the ∞-category $\mathrm{Atom}_{\mathcal{O}}(n)$ is equivalent to the classifying space $\mathrm{B}\mathcal{O}(1)$ of the group of automorphisms of $\langle 1 \rangle$ in $\mathcal{O}$.*

Proof of lemma 5.3.3. First, fix an atomic morphism $\alpha\colon \langle n\rangle \to \langle n+1\rangle$ in $\mathbb{F}_*$ and consider the subcategory $\mathrm{Atom}^{\alpha}_{\mathcal{O}}(n)$ of morphisms lying over α. By definition, we have a cartesian square

$$\begin{tikzcd} \mathrm{Atom}^{\alpha}_{\mathcal{O}}(n) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Atom}_{\mathcal{O}}(n) \arrow[d] \\ \{\alpha\} \arrow[r] & \mathrm{Atom}_{\mathbb{F}_*}(n). \end{tikzcd}$$

Observe that the ∞-category $\mathrm{Atom}_{\mathbb{F}_*}(n)$ is the nerve of a 1-category in which any two objects $j\colon \langle n\rangle \to S$ and $j'\colon \langle n\rangle \to S'$ are related by a unique morphism $S \to S'$ (namely the unique bijection that restricts to $j' \circ (j|^{\mathrm{im}(j)})^{-1}$ on the image of j). As a consequence, this (∞-)category is terminal and we obtain a canonical equivalence of ∞-categories $\mathrm{Atom}^{\alpha}_{\mathcal{O}}(n) \simeq \mathrm{Atom}_{\mathcal{O}}(n)$.

Now we may decompose the atomic morphisms of $\mathrm{Atom}^{\alpha}_{\mathcal{O}}(n)$ according to their arity using lemma 3.5.3. The result is an equivalence of ∞-categories

$$\mathrm{Atom}^{\alpha}_{\mathcal{O}}(n) \simeq \left(\mathcal{O}_{\langle 1\rangle/}\right)^n \times \mathcal{O}_{\langle 0\rangle/}$$

where the ∞-category $\mathcal{O}_{\langle 0\rangle/}$ is a notation for the comma category

$$(\langle 0\rangle \in \mathcal{O}^{\otimes}) \downarrow (\mathcal{O}^{\otimes} \supset \mathcal{O}).$$

Since $\mathcal{O}$ is assumed to be an ∞-groupoid, so is its slice $\mathcal{O}_{\langle 1\rangle/}$; the latter has an initial object, it is therefore contractible. We now turn to analysing the comma category $\mathcal{O}_{\langle 0\rangle/}$. By definition, it fits in a commutative diagram of cartesian squares

$$\begin{tikzcd} \mathcal{O}_{\langle 0\rangle/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & (\mathcal{O}^{\otimes})^{\langle 0\rangle/} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Fun}(\Delta^1, \mathcal{O}^{\otimes}) \arrow[d, "(\mathrm{ev}_0,\mathrm{ev}_1)"] \\ \{\langle 0\rangle\} \times \mathcal{O} \arrow[r, hook] & \{\langle 0\rangle\} \times \mathcal{O}^{\otimes} \arrow[r, hook] & \mathcal{O}^{\otimes} \times \mathcal{O}^{\otimes}. \end{tikzcd}$$

Since the middle vertical map is a cocartesian fibration, so is the left vertical map. The fiber of this morphism at at object $X \in \mathcal{O}$ is $\mathrm{Map}_{\mathcal{O}^{\otimes}}(\langle 0\rangle, X)$, which is contractible by the assumption that $\mathcal{O}^{\otimes}$ is unital. Therefore this cocartesian fibration is a trivial fibration $\mathcal{O}_{\langle 0\rangle/} \xrightarrow{\simeq} \mathcal{O}$, which completes the proof. $\square$

Proof of Theorem 5.1.1. The homotopy cartesian square (5.1) is obtained as the top left square in diagram (5.8), using the equivalences $\mathrm{Ext}^{\triangle}(\sigma) \simeq \mathrm{Ext}^{\square}(\sigma) \simeq \mathrm{Ext}(\sigma)$ of lemmas 5.3.2 and 5.2.5 and the equivalence $\mathrm{Atom}^{\sigma}_{\mathcal{O}}(n) \simeq \mathrm{B}\mathcal{O}(1)$ of corollary 5.3.4. $\square$

5.4 Applications

As explained in the introduction (section 1.3), an important motivation for studying the brane action comes from string topology, as the $\mathbb{E}_2$-algebra structure on free loop spaces arises from span diagrams given the brane action for the ∞-operad $\mathbb{E}_2$. More precisely, recall from the program described in section 1.3 that our work was motivated by the desire to use the formalism of brane actions to generalize string topology in the following directions.

- On the one hand, we may consider analogs of free loop spaces $\mathrm{Map}(S^{n-1}, X)$ based on higher dimensional spheres (brane topology).

- On the other hand, with an eye towards conjecture 1.2.1, we would like to enhance the $\mathbb{E}_n$-structure of brane topology to take into account the action of groups of homeomorphisms of disks. The ∞-operads governing such structures are variants $\mathbb{E}_n^G$ of the little disks ∞-operad $\mathbb{E}_n$, that are given by semi-direct product of $\mathbb{E}_n$ with a group G endowed with a morphism to the ∞-group $\mathsf{Top}(n)$ associated to the topological group of self-homeomorphisms of $\mathbb{R}^n$.

Remark 5.4.1. As noticed when discussing program 1.3 in the introduction, although the application of the formalism of brane actions to brane topology (i.e. the first of the above directions of generalization) is already possible using the original results of [Toë13], the latter generalization requires to extend the formalism of brane actions to coherent ∞-operads whose space of unary operations may not be trivial, a problem that has been adressed in the previous chapters with theorem A.

The rest of this chapter is devoted to the study of a generalization of the ∞-operads $\mathbb{E}_n^G$, a proof of their coherence and, as a consequence, a construction of new operations operations on spaces of branes in a derived stack.

5.4.1 Coherence of the little B-framed disks ∞-operad

In this section, we define the ∞-operad of B-framed little disks and prove that it is coherent. This ∞-operad depends on the datum of a Kan fibration $B \to \mathrm{B}\mathsf{Top}(n)$ and recovers the variants $\mathbb{E}_n^G$ of the little disks ∞-operad mentionned above when B is the classifying space of a subgroup G of $\mathsf{Top}(n)$.

We recall the definition of ∞-operad $\mathbb{E}_B^{\otimes}$ introduced in [Lur17, Section 5.4.2], following the presentation and the notations thereof.

Notation 5.4.2. Given two topological spaces X and Y, we let $\mathrm{Emb}(X, Y)$ denote the topological space of open embeddings $X \to Y$, topologized as a subspace of the compact-open topology on the set $\mathrm{Hom}_{\mathsf{Top}}(X, Y)$. For $n \in \mathbb{N}$, we let $\mathsf{Top}(n)$ denote the topological space of homeomorphisms of $\mathbb{R}^n$, viewed as a subspace of $\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$.

Remark 5.4.3. The Kister–Mazur theorem implies that the inclusion $\mathsf{Top}(n) \to \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)$ is a homotopy equivalence, for all $n \geqslant 0$ (see [Lur17, Theorem 5.4.1.5]).

Let us fix a natural number n.

Definition 5.4.4 ([Lur17, Definition 5.4.2.1])**.** Let ${}^t\mathbb{E}^{\otimes}_{\mathrm{B}\mathsf{Top}(n)}$ denote the topological category whose objects are the finite pointed sets $\langle m\rangle \in \mathbb{F}_*$ and where mapping spaces are given by the formula

$$\mathrm{Map}_{{}^t\mathbb{E}^{\otimes}_{\mathrm{B}\mathsf{Top}(n)}}(\langle m\rangle, \langle k\rangle) = \coprod_{\alpha\in \mathrm{Hom}_{\mathbb{F}_*}(\langle m\rangle,\langle k\rangle)} \prod_{i=1}^{m} \mathrm{Emb}(\mathbb{R}^n \times \alpha^{-1}\{i\}, \mathbb{R}^n). \qquad (5.10)$$

Let $\mathrm{B}\mathsf{Top}(n)^{\otimes}$ denote its homotopy coherent nerve, i.e. the ∞-category $N\left({}^t\mathbb{E}^{\otimes}_{\mathrm{B}\mathsf{Top}(n)}\right)$. By [Lur17, Proposition 2.1.1.27], $\mathrm{B}\mathsf{Top}(n)^{\otimes}$ forms an ∞-operad. We will denote by $\mathrm{B}\mathsf{Top}(n)$ its underlying ∞-category, which by remark 5.4.3 is a classifying space for the topological group $\mathsf{Top}(n)$.

Let us fix a Kan complex B together with a Kan fibration $B \to \mathrm{B}\mathsf{Top}(n)$.

Notation 5.4.5. Recall that given an ∞-category $\mathcal{C}$, one can construct a cocartesian ∞-operad $\mathcal{C}^{\amalg}$ whose spaces of multimorphisms are given by the formula $\mathrm{Mul}_{\mathcal{C}^{\amalg}}(c_1, \dots, c_m; c) = \prod_{i=1}^m \mathrm{Map}_{\mathcal{C}}(c_i, c)$ (see [Lur17, Section 2.4.3]).

Definition 5.4.6 ([Lur17, Definition 5.4.2.10])**.** We let $\mathbb{E}^{\otimes}_B$ denote the ∞-operad

$$\mathbb{E}^{\otimes}_B = \mathrm{B}\mathsf{Top}(n)^{\otimes} \underset{\mathrm{B}\mathsf{Top}(n)^{\amalg}}{\times} B^{\amalg} \qquad (5.11)$$

and refer to it as the ∞*-operad of B-framed little disks.*

Note that the underlying ∞-category of $\mathbb{E}^{\otimes}_B$ is canonically equivalent to the Kan complex B. In particular, one may identify the objects of $\mathbb{E}^{\otimes}_B$ with those of B.

Remark 5.4.7 (Examples).

- For B a contractible Kan complex with a Kan fibration to $\mathrm{B}\mathsf{Top}(n)$, the associated ∞-operad $\mathbb{E}^{\otimes}_B$ reduces to the ordinary ∞-operad $\mathbb{E}^{\otimes}_n$ of little disks of dimension n.

- Consider a topological group together with a map to $\mathsf{Top}(n)$. The induced morphism on classifying space can be represented up to equivalence by a Kan fibration $B := \mathrm{B}G \to \mathrm{B}\mathsf{Top}(n)$. Then the ∞-operad of B-framed little disks $\mathbb{E}^{\otimes}_B$ is equivalent to a semi-direct $\mathbb{E}^{\otimes}_n \rtimes G$ (in the sense of [SW03]). As a particular case, for $G = SO(n)$ we obtain the framed little disks ∞-operad $\mathbb{E}^{\mathrm{fr}}_n$.

- Let M be a topological manifold of dimension n. Following [Lur17, Definition 5.4.5.1.], let $\mathcal{C}_M$ denote the topological category with two objects M and $\mathbb{R}^n$, whose mapping spaces are

$$\begin{aligned} &\mathrm{Map}_{\mathcal{C}_M}(\mathbb{R}^n, \mathbb{R}^n) = \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n) && \mathrm{Map}_{\mathcal{C}_M}(\mathbb{R}^n, M) = \mathrm{Emb}(\mathbb{R}^n, M) \\ &\mathrm{Map}_{\mathcal{C}_M}(M, \mathbb{R}^n) = \emptyset && \mathrm{Map}_{\mathcal{C}_M}(M, M) = \{\mathrm{id}_M\}. \end{aligned}$$

Define B_M as the Kan complex $\mathrm{B}\mathsf{Top}(n) \times_{\mathcal{N}(\mathcal{C}_M)} \mathcal{N}(\mathcal{C}_M)_{/M}$ and let $\mathbb{E}^{\otimes}_M$ denote the ∞-operad $\mathbb{E}^{\otimes}_{B_M}$. It is a variant on the ∞-operad $\mathbb{E}^{\otimes}_n$ in which colors are open embedding $U\colon \mathbb{R}^n \to M$ of disks of dimension n into M and operations of arity k are diagrams of embeddings

$$\begin{tikzcd} \coprod_{i=1}^k \mathbb{R}^n \arrow[rr] \arrow[dr, "\coprod_i U_i"'] & & \mathbb{R}^n \arrow[dl, "U"] \\ & M & \end{tikzcd}$$

together with an isotopy making the triangle commute. Note that $\mathbb{E}_M$-algebras can be identified as locally constant factorization algebras on M, by theorem [Lur17, Theorem 5.4.5.9.].

The ∞-operad of little disks operad $\mathbb{E}_n$ is coherent, for all $n \in \mathbb{N}$ by [Lur17, Theorem 5.1.1.1]. We generalize this result to the B-framed situation.

Theorem 5.4.8. *The ∞-operad of B-framed little disks $\mathbb{E}_B^{\otimes}$ is coherent.*

The proof relies on theorem 5.1.1 together with the following computation.

Lemma 5.4.9. *Let $\sigma\colon (b_1, \dots, b_m) \to b$ be an active morphism in $\mathbb{E}_B^{\otimes}$, with $b_1, \dots, b_m, b$ in B and choose an additional color b_{m+1} in B. Then Toën's model for the space of extensions of σ in $\mathbb{E}_B^{\otimes}$ is given by*

$$\mathrm{Mul}_{\mathbb{E}_B}(b_1, \dots, b_{m+1}; b) \underset{\mathrm{Mul}_{\mathbb{E}_B}(b_1,\dots,b_m;b)}{\overset{\mathrm{h}}{\times}} \{\sigma\} \simeq \begin{cases} \Omega_b B \times \bigvee^m S^{n-1} & \text{if } b_{m+1} \simeq b \text{ in } B, \\ \emptyset & \text{otherwise.} \end{cases} \tag{5.12}$$

Proof. By construction, the left hand side of equation (5.12) is equivalent to the homotopy fiber at σ of the map

$$\begin{array}{c} \mathrm{Emb}(\mathbb{R}^n \times \langle m+1 \rangle^\circ, \mathbb{R}^n) \underset{\mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n)^{m+1}}{\times} \prod_{i=1}^{m+1} \mathrm{Map}_B(b_i, b) \\ \downarrow \\ \mathrm{Emb}(\mathbb{R}^n \times \langle m \rangle^\circ, \mathbb{R}^n) \underset{\mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n)^{m}}{\times} \prod_{i=1}^{m} \mathrm{Map}_B(b_i, b). \end{array} \tag{5.13}$$

Commuting the fiber product with the homotopy fiber, we obtain the space

$$\left(\mathrm{Emb}(\mathbb{R}^n \times \langle m+1 \rangle^\circ, \mathbb{R}^n) \underset{\mathrm{Emb}(\mathbb{R}^n \times \langle m \rangle^\circ, \mathbb{R}^n)}{\overset{\mathrm{h}}{\times}} \{\sigma\} \right) \underset{\mathrm{Emb}(\mathbb{R}^n,\mathbb{R}^n)}{\times} \mathrm{Map}_B(b_{m+1}, b) \tag{5.14}$$

Since B is a Kan complex, the factor $\mathrm{Map}_B(b_{m+1}, b)$ is empty when b_{m+1} and b are in different connected components, and is equivalent to the based loop space $\Omega_b B$ otherwise. On the other hand, for any finite set S, the obvious map from the space $\mathrm{Emb}(\mathbb{R}^n \times S, \mathbb{R}^n)$ to the product $\mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n)^S \times \mathrm{Conf}(S, \mathbb{R}^n)$ is an equivalence (see [Lur17, Proof of Proposition 5.4.2.8.]). As a consequence, we obtain an equivalence

$$\mathrm{Emb}(\mathbb{R}^n \times \langle m+1 \rangle^\circ, \mathbb{R}^n) \underset{\mathrm{Emb}(\mathbb{R}^n \times \langle m \rangle^\circ, \mathbb{R}^n)}{\overset{\mathrm{h}}{\times}} \{\sigma\} \simeq \mathrm{Emb}(\mathbb{R}^n, \mathbb{R}^n) \times \mathrm{Conf}(S, \mathbb{R}^n). \tag{5.15}$$

Substituting this equivalence in (5.15) and using that $\mathrm{Conf}(S, \mathbb{R}^n) \simeq \bigvee^m S^{n-1}$, we obtain the result. $\square$

Lemma 5.4.10. *Let $\sigma\colon (b_1,\dots,b_m) \to b$ be an active morphism in $\mathbb{E}_B^{\otimes}$, with $b_1,\dots,b_m,b$ in B. Then the space of extensions of σ in $\mathbb{E}_B^{\otimes}$ is equivalent to $\bigvee^m S^{n-1}$.*

Proof. By theorem 5.1.1, any choice of a color b_{m+1} yields a homotopy cartesian square

$$\begin{tikzcd} \mathrm{Mul}_{\mathbb{E}_B}(b_1,\dots,b_{m+1};b) \overset{\mathrm{h}}{\underset{\mathrm{Mul}_{\mathbb{E}_B}(b_1,\dots,b_m;b)}{\times}} \{\sigma\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Ext}(\sigma) \arrow[d] \\ \{b_{m+1}\} \arrow[r] & B, \end{tikzcd} \tag{5.16}$$

Upon taking base change of $\mathrm{Ext}(\sigma) \to B$ along the inclusion $B_{[b]} \to B$ of the connected component of b and using proposition A.3.8, square (5.16) endows the space of strict extensions (at the top left corner of equation (5.16)) with an $\Omega_b B$-principal ∞-bundle structure over $\mathrm{Ext}(\sigma)$.

If b_{m+1} does not belong to the connected component of b in B, then the corresponding fiber of $\mathrm{Ext}(\sigma)_{[b]}$ over $B_{[b]}$ is empty. In particular, the structural map $\mathrm{Ext}(\sigma) \to B$ factors through $B_{[b]}$. Now choose a point $b_{m+1} \in B_{[b]}$. Through the identification given by lemma 5.4.9, the $\Omega_b B$-action on the space of strict extensions is the regular action on the first factor of $\Omega_b B \times \bigvee^m S^{n-1}$. Taking the quotient by this action, we see that the space $\mathrm{Ext}(\sigma)$ is equivalent to $\bigvee^m S^{n-1}$. □

Proof of theorem 5.4.8. First, it is clear that the ∞-operad $\mathbb{E}_B^{\otimes}$ is unital. Moreover, its underlying ∞-category B is a Kan complex by assumption. It remains to prove condition (c) of definition 2.2.6. By lemma 5.4.10, for a sequence of composable active morphisms $X \xrightarrow{\sigma} Y \xrightarrow{\tau} Z$ in $\mathbb{E}_B^{\otimes}$, with X, Y and Z of arity respectively m, k and 1, diagram (2.4) is equivalent in the homotopy category of spaces to a commutative square of the form

$$\begin{tikzcd} \coprod^{k} S^{n-1} \arrow[r] \arrow[d] & \bigvee^k S^{n-1} \arrow[d] \\ \coprod_{i=1}^{k} \bigvee_{p(\sigma)^{-1}\{i\}} S^{n-1} \arrow[r] & \bigvee^{m} S^{n-1} \end{tikzcd} \tag{5.17}$$

which is easily seen to be homotopy cocartesian (as in the case of the little disks ∞-operad $\mathbb{E}_n$). This concludes the proof. □

5.4.2 Action of the little B-framed disks ∞-operad on spaces of branes

As we just established, the ∞-operad $\mathbb{E}_B^{\otimes}$ is coherent (theorem 5.4.8); therefore, it admits a brane action by theorem A. This yields the following result.

Corollary 5.4.11. *Using the same notations as above, there is a canonical morphism of ∞-operads $\mathbb{E}_B^{\otimes} \to \mathrm{Cospan}(\mathcal{S})^{\amalg}$, sending a color b to the space $\mathrm{Ext}(\mathrm{id}_b) \simeq S^{n-1}$ and an operation $\sigma\colon (b_1, \dots, b_m) \to b$ to a cospan diagram*

$$\mathrm{Ext}(\mathrm{id}_b)^{\amalg m} \longrightarrow \mathrm{Ext}(\sigma) \longleftarrow \mathrm{Ext}(\mathrm{id}_b) \tag{5.18}$$

in which the middle space $\mathrm{Ext}(\sigma)$ is equivalent to a wedge of m spheres S^{n-1}.

As explained in section 1.3, the previous result can be applied to multiple geometric contexts. Let us recall the method to produce operations on spaces of branes.

Let $\mathcal{X}$ be an ∞-topos. There is a canonical functor $(-)_{\mathrm{cst}}\colon \mathcal{S} \to \mathcal{X}$ that sends every space Z to the object Z_{cst} obtained as the colimit of the constant diagram with shape Z and value the terminal object $*$. One could call Z_{cst} the locally constant stack with value Z, or the Betti shape of Z in $\mathcal{X}$. Through this functor, we obtain from the $\mathbb{E}_B$-algebra structure of Corollary 5.4.11 a corresponding algebra in the ∞-category $\mathrm{Cospan}(\mathcal{X})$.

Given an object $X \in \mathcal{X}$, we may now apply the functor $\mathrm{Map}(-, X)\colon \mathcal{X}^{\mathrm{op}} \to \mathcal{X}$ to the objects $\mathrm{Ext}(\sigma)_{\mathrm{cst}}$ to obtain an $\mathbb{E}_B$-algebra structure in $\mathrm{Span}(\mathcal{X})$. To summarize, we have the following result.

Corollary 5.4.12. *For any object X in $\mathcal{X}$, the space $\mathrm{Map}((S^{n-1})_{\mathrm{cst}}, X)$ of $\mathbb{E}_B$-branes internal to $\mathcal{X}$ has a canonical $\mathbb{E}_B$-algebra structure in $\mathrm{Span}(\mathcal{X})$, with structural morphism sending an operation σ of arity m to the span*

$$\mathrm{Map}((S^{n-1})_{\mathrm{cst}}, X)^m \longleftarrow \mathrm{Map}(\mathrm{Ext}(\sigma)_{\mathrm{cst}}, X) \longrightarrow \mathrm{Map}((S^{n-1})_{\mathrm{cst}}, X). \tag{5.19}$$

Remark 5.4.13. The advantage of the above construction is its generality: one can say that the $\mathbb{E}_B$-algebra structure on $\mathrm{Map}((S^{n-1})_{\mathrm{cst}}, X)$ in $\mathrm{Span}(\mathcal{X})$ is motivic, in the sense that it exists before taking any sort of linear invariant (chains, cohomology, quasi-coherent shaves, K-theory, etc.).

This is similar to the case of Gromov–Witten invariants [MR18], where the authors use the brane action to construct Gromov–Witten invariants at a purely geometric (or motivic) level and are then able to apply K-theory or ordinary cohomology functors to recover the invariants in their more classical form.

In particular, specializing to the case $B = \mathrm{B}SO(n)$, the ∞-operad $\mathbb{E}_B^{\otimes}$ recovers that of framed little disks $\mathbb{E}_n^{\mathrm{fr}}$, so that the above corollary gives the following partial answer to conjecture 1.2.1.

Corollary 5.4.14. *Let X be a topological space. Then the brane space $\mathrm{Map}(S^{n-1}, X)$ carries an $\mathbb{E}_n^{\mathrm{fr}}$-algebra structure in $\mathrm{Span}(\mathcal{S})$.*

Inverting spans

In many applications, it is useful to "invert" the wrong-way morphisms appearing in the spans to obtain an algebra structure in a more tractable ∞-category, such as that of chain complexes or of spectra. To make this construction more precise, we rely on the universal property of the category of spans, as established in [Ste20] (see also [GR17] for an earlier description of this universal property).

First, given an ∞-category $\mathcal{C}$ with pullbacks, we consider an $(\infty,2)$-enhancement $\mathsf{Span}_2(\mathcal{C})$ of the ∞-category $\mathrm{Span}(\mathcal{C})$ of spans in $\mathcal{C}$ (see [Hau18] or [Ste20] for a precise construction).

Next, we define the Beck–Chevalley condition.

Definition 5.4.15 (Adjointable squares, [Ste20, Definition 3.4.1])**.** Let D be an $(\infty,2)$-category. A commutative square

$$\begin{tikzcd} d' \arrow[r, "\alpha'"] \arrow[d, "\beta'"] & d \arrow[d, "\beta"] \\ e' \arrow[r, "\alpha"] & e \end{tikzcd} \tag{5.20}$$

in D is called *vertically right adjointable* if β and β' admit right adjoints β^R and β'^R and moreover the canonical 2-morphism

$$\alpha'\beta'^R \to \beta^R\alpha \tag{5.21}$$

constructed using the unit $\mathrm{id}_d \to \beta^R\beta$ and the counit $\beta'^R\beta' \to \mathrm{id}_{e'}$, is an isomorphism.

The square is said to be *horizontally right adjointable* if its transpose is vertically right adjointable. If it is both vertically and horizontally right adjointable, we simply say that the square is *right adjointable.*

Definition 5.4.16 (Beck–Chevalley condition, [Ste20, Definition 3.4.5])**.** Let $\mathcal{C}$ be an ∞-category with pullbacks and D be an $(\infty,2)$-category. A functor $F\colon \mathcal{C} \to \mathsf{D}$ is said to satisfy the *left Beck–Chevalley* condition if for every cospan $x \to s \leftarrow y$ in $\mathcal{C}$, the induced commutative square in $\mathcal{D}$

$$\begin{tikzcd} F(x \times_s y) \arrow[r] \arrow[d] & F(y) \arrow[d] \\ F(x) \arrow[r] & F(s) \end{tikzcd}$$

is right adjointable.

Using these definitions, one can characterize the 2-functors out of $(\infty,2)$-categories of spans.

Theorem 5.4.17 (2-categorical universal property of spans, [Ste20, Theorem 3.4.18])**.** *Let $\mathcal{C}$ be an ∞-category with pullbacks and D be an $(\infty,2)$-category. Precomposition with the canonical functor $\mathcal{C} \to \mathsf{Span}_2(\mathcal{C})$ identifies the space of 2-functors $\mathsf{Span}_2(\mathcal{C}) \to \mathsf{D}$ with the subspace of $\mathrm{Map}_{\mathsf{Cat}_\infty}(\mathcal{C},\mathsf{D})$ consisting of those functors $\mathcal{C} \to \mathsf{D}$ that satisfy the left Beck–Chevalley condition.*

Remark 5.4.18. Given a functor $F\colon \mathcal{C} \to \mathsf{D}$ satisfying the left Beck–Chevalley condition, the associated 2-functor $\mathsf{Span}_2(\mathcal{C}) \to \mathsf{D}$ sends a span $x \xleftarrow{f} s \xrightarrow{g} y$ to the morphism $f^R \circ g\colon F(x) \to F(y)$.

We briefly discuss two geometric contexts: the case of derived stacks $\mathcal{X} = \mathrm{d}\mathcal{S}\mathrm{t}_k$ and that of spaces $\mathcal{X} = \mathcal{S}$.

Algebro-geometric context. Let k be a field of characteristic 0 and $\mathrm{d}\mathcal{S}\mathrm{t}_k$ the ∞-category of derived étale stacks over k.

To invert the correspondences of derived stacks that arise from the brane action, we need to restrict our attention to a particularly well-behaved class of spaces, namely that of perfect stacks introduced by Ben-Zvi–Francis–Nadler in [BZFN10].

Definition 5.4.19 (Perfect stacks, [BZFN10])**.** A derived stack X is said to be *perfect* if its diagonal morphism is affine and if $\mathrm{QCoh}(X)$ is the ind-completion of its full subcategory of perfect complexes. We let $\mathcal{P}$ denote the full subcategory of $\mathrm{d}\mathcal{S}\mathrm{t}_k$ on perfect stacks.

Example 5.4.20. The class of perfect stacks contain many examples of interest. For instance, every quotient Y/G of a quasi-projective derived scheme Y by a linear action of an affine group G is perfect. Perfect stacks are moreover stable under fiber products and if $X \in \mathcal{P}$, so is $\mathrm{Map}((K)_{\mathrm{cst}}, X)$ for every finite simplicial set K.

Consider the $(\infty, 2)$-category $\mathrm{dg}\mathcal{C}\mathrm{at}_k^{\mathrm{L}}$ of (possibly large) k-linear presentable dg-categories, with functors preserving small colimits as morphisms. Let $\mathrm{QCoh}\colon \mathrm{d}\mathcal{S}\mathrm{t}_k \to \mathrm{dg}\mathcal{C}\mathrm{at}_k^{\mathrm{L}}$ denote the functor that assigns to every derived stack its derived ∞-category of quasi-coherent sheaves.

By [BZFN10, Proposition 3.10], the restriction of QCoh to $\mathcal{P}$ satisfies the left Beck–Chevalley property and therefore extends to a 2-functor

$$\mathrm{QCoh}\colon \mathsf{Span}_2(\mathcal{P}) \to \mathrm{dg}\mathcal{C}\mathrm{at}_k^{\mathrm{L}},$$

using Theorem 5.4.17. Moreover, we can upgrade this 2-functor QCoh to a symmetric monoidal one, using [Ste20, Corollary 1.2.2]. Together with Corollary 5.4.12, this shows the following result.

Corollary 5.4.21. *Let X be a perfect stack. Then the $(\infty, 2)$-category of quasicoherent sheaves on its space of branes $\mathrm{Map}((S^{n-1})_{\mathrm{cst}}, X)$ carries a canonical $\mathbb{E}_B$-algebra structure in $\mathrm{dg}\mathcal{C}\mathrm{at}_k^{\mathrm{L}}$.*

This results extends results of Toën [Toë13, Corollary 5.1] and of Ben-Zvi–Francis–Nadler [BZFN10], which corresponds to the particular case of the $\mathbb{E}_n$-operad (that is $\mathbb{E}_B$ for $B \simeq *$).

Topological context. We may want to adapt the above construction to the case of topological spaces, in order to recover the classical string topology operations, and more generally to prove conjecture 1.2.1 (this was essentially Program 1.3 from the introduction).

However, one immediately runs into the problem of defining *functorial umkehr (or wrong-way) maps at the chain level.* In particular, this would require to handle all the coherence data needed to produce a functor from a suitable subcategory $\mathcal{P}$ of $\mathcal{S}$ to that of chain complexes (or suitable variants of such). The subcategory $\mathcal{P}$ would have to contain the free loop spaces $\mathcal{L}X$, which are infinite-dimensional manifolds, and the sought functor to chain complexes would have to specialize to the classical Thom–Pontryagin construction of umkehr maps upon taking homology.

To the knowledge of the author, the existence of such a construction is still an open question.

Appendix A

Recollections

A.1 Marked simplicial sets

In this section, we collect various facts about marked simplicial sets that are used in the proof of theorem B. These results are standard and well-known to specialists, with perhaps the exception of proposition A.1.7, stating that anodyne morphisms satisfy a weak form of the right simplification property, which seems to have not appeared in the literature. This last result might be of independent interest.

A.1.1 Some properties of marked simplicial sets

Definition A.1.1. A *marked simplicial set* is a pair (X, mX) where X is a simplicial set and mX is a subset of X_1 that contains all degenerate edges. A morphism of marked simplicial sets $(X, mX) \to (Y, mY)$ is a morphism of simplicial sets $f\colon X \to Y$ such that $f(mX) \subseteq mY$.

The category of marked simplicial sets is denoted sSet^+.

Notation A.1.2. Given a simplicial set X, one can associated three marked simplicial sets:

- the minimal marking $X^\flat$, consisting only of degenerate edges,
- the cartesian marking $X^\natural$ in which an edge is marked if and only if it is an equivalence,
- the maximal marking $X^\sharp$, containing all edges.

Given an edge e in a marked simplicial set Y, we let $Y[e]$ denote the marked simplicial set obtained by further marking e. In other words, $Y[e]$ is the initial marked simplicial set whose underlying simplicial set is Y and such that both canonical maps $Y \to Y[e]$ and $e\colon \Delta^{1,\sharp} \to Y[e]$ are morphisms of marked simplicial sets.

Given a category $\mathcal{C}$ and a class S of morphisms in $\mathcal{C}$, we say that S is *weakly saturated* if it contains all isomorphisms and is closed under cobase change, transfinite composition, coproducts and retracts. The smallest weakly saturated class containing S is denoted $\overline{S}$ and called the *saturation* of S.

We introduce several classes of morphisms in sSet and sSet^+:

- the class $\mathrm{Cell} = \{\partial\Delta^n \subset \Delta^n \mid n \in \mathbb{N}\}$,
- the class $\mathrm{InnHorn} = \{\Lambda^n_k \subset \Delta^n \mid 0 < k < n,\ n \geqslant 2\}$,
- the class $\mathrm{Cell}^\flat = \{\Lambda^{n,\flat}_k \subset \Delta^{n,\flat} \mid 0 \leqslant k < n,\ n \geqslant 1\}$,
- the class $\mathrm{InnHorn}^\flat = \{\Lambda^{n,\flat}_k \subset \Delta^{n,\flat} \mid 0 < k < n,\ n \geqslant 2\}$,
- the class $\mathrm{LHorn}^\sharp = \mathrm{InnHorn}^\flat \cup \{\Lambda^{n,\flat}_0[0 \to 1] \subset \Delta^{n,\flat}[0 \to 1] \mid n \in \mathbb{N}^*\}$,
- the class $\mathrm{RHorn}^\sharp = \mathrm{InnHorn}^\flat \cup \{\Lambda^{n,\flat}_n[n-1 \to n] \subset \Delta^{n,\flat}[n-1 \to n] \mid n \in \mathbb{N}^*\}$.

The saturations $\overline{\mathrm{Cell}}$ and $\overline{\mathrm{InnHorn}}$ are respectively the class of monomorphisms and that of inner anodyne morphisms. We now introduce a notion of anodyne morphisms for marked morphisms that is suitable for our computations of chapter 4.

Definition A.1.3 (Marked anodyne morphisms)**.** The class Mark of marked anodyne morphisms is defined as the saturation of the union of $\mathrm{LHorn}^\sharp$ and $\mathrm{RHorn}^\sharp$ together with the map

$$\Lambda^{2,\sharp}_1 \coprod_{\Lambda^{2,\flat}_1} \Delta^{2,\flat} \longrightarrow \Delta^{2,\sharp}$$

as well as the maps $K^\flat \to K^\sharp$ for all Kan complexes K.

Remark A.1.4 (Difference with Lurie's definition)*.* Beware that the previous definition differs from that [Lur09a, Definition 3.1.1.1] in that our definition is symmetric, whereas Lurie's include $\mathrm{RHorn}^\sharp$ but not $\mathrm{LHorn}^\sharp$. The conceptual reason for this discrepancy is the following: Lurie's marked anodyne morphisms are examples of trivial cofibration in the cartesian model structure on sSet^+, while our marked anodyne morphisms should be trivial cofibrations in an appropriate model structure of bifibrations on sSet^+. However, for the purpose of this work, we shall not need the full power of such a model structure.

Definition A.1.5. Morphisms satisfying definition [Lur09a, Definition 3.1.1.1] will be called *marked right anodyne* in this thesis. The obvious dual definition gives the class of *marked left anodyne* morphisms.

Lemma A.1.6. *Every marked anodyne morphism has the left lifting property against all morphisms of the form $X^\natural \to *$ for X an ∞-category.*

Proof. By [Lur09a, Proposition 3.1.1.6], marked *right* anodyne morphisms have the desired lifting property. By symmetry of the argument, so do marked left anodyne morphisms. Since Mark is the saturation of the class given as the union of these two types of anodyne morphisms, we deduce the result. □

In chapter 4, we will use that the class of marked anodyne morphisms satisfy the following weak form of right cancellation property.

Proposition A.1.7 (Right cancellation property for marked anodyne morphisms)**.** *Let $i\colon A \to B$ and $j\colon B \to C$ be monomorphisms of marked simplicial sets. Assume that i and $j \circ i$ are marked anodyne morphisms and that j is bijective on 0-simplices. Then j has the left lifting property with respect to all morphisms of the form $X^\natural \to *$ for X an ∞-category.*

Proof. Our proof is merely an adaptation to the marked simplicial setting of the argument of [Ste18, Theorem 1.5] which states that the class of inner anodyne maps has the right cancellation property. We give details here for completeness.

Let X be an ∞-category. We will show that j has the left lifting property against $X^\natural \to *$. Suppose we are given a map $u\colon B \to X$ of marked simplicial sets. By lemma A.1.6, that $j \circ i$ is marked anodyne allows to pick a morphism $\varphi\colon C \to X$ satisfying $\varphi \circ j \circ i = u \circ i$. This implies that u and $\varphi \circ j$ are in the same fiber of the map $i^*\colon \mathrm{Map}^\sharp(B, X^\natural) \longrightarrow \mathrm{Map}^\sharp(A, X^\natural)$. By [Lur09a, Proposition 3.1.3.3 and the following remark] (or more precisely a generalization thereof to arbitrary marked anodyne maps in our sense), the map i^* is a trivial Kan fibration. We may then take a homotopy between u and $\varphi \circ j$ over their common image by i^*. This homotopy takes the form of a morphism $h\colon \Delta^{1,\sharp} \times B \to X^\natural$ with the following properties:

$$h|_{\{0\}\times B} = \varphi \circ j \qquad h|_{\{1\}\times B} = u \qquad h \circ (\mathrm{id}_A \times i) = u \circ i \circ \mathrm{proj}_A.$$

Consequently, h and φ induce a map $w = (\varphi, h)\colon \{0\} \times C \cup \Delta^{1,\sharp} \times B \to X^\natural$. The problem therefore reduces to finding a lift $d\colon \Delta^{1,\sharp} \times C \to X^\natural$ in the diagram

$$\begin{tikzcd}
\{0\} \times C \cup \Delta^{1,\sharp} \times B \arrow[r, "w"] \arrow[d] & X^\natural \\
\Delta^{1,\sharp} \times C \arrow[ur, dashed, bend right, "d"] &
\end{tikzcd}$$

for then $d|_{\{1\}\times C}$ will provide the desired lift of u along j. Using the skeleton filtration on C, write $C(n) = B \cup \mathrm{sk_n(C)}$. Note that we have the equality $B = C(0)$, since j is bijective on objects. Working inductively, it therefore suffices to prove the existence of a lift in the following diagram:

$$\begin{tikzcd}
(\{0\} \times C(n+1)) \cup (\Delta^{1,\sharp} \times C(n)) \arrow[r] \arrow[d] & X^\natural \\
\Delta^{1,\sharp} \times C(n+1), \arrow[ur, dashed, bend right] &
\end{tikzcd}$$

for every $n \in \mathbb{N}$. Since every monomorphism of marked simplicial sets is obtained by cell attachments and edge markings, the proof reduces to the case where the inclusion $C(n) \to C(n+1)$ is either $\partial\Delta^{n+1,\flat} \to \Delta^{n+1,\flat}$ or $\Delta^{1,\flat} \to \Delta^{1,\sharp}$. Using [Lur09a, Corollary 3.1.1.7], one easily sees that $(\Delta^{1,\sharp} \times \Delta^{1,\flat}) \cup (\{0\} \times \Delta^{1,\sharp}) \to \Delta^{1,\sharp} \times \Delta^{1,\sharp}$ is marked anodyne, so the result follows for the case of the latter inclusion. For the former inclusion, one can adapt the argument of [Ste18, Lemma 2.4]: decompose the inclusion

$$(\{0\} \times \Delta^{n+1,\flat}) \cup (\Delta^{1,\sharp} \times \partial\Delta^{n,\flat}) \longrightarrow \Delta^{1,\sharp} \times \Delta^{n+1,\flat}$$

as a sequence of inner horn inclusions that successively add the different top-dimensional simplices, composed with the inclusion of a left marked horn $\Lambda_0^{n+1}[0 \to 1]$ into $\Delta^{n+1}[0 \to 1]$. □

A.1.2 Calculus of pushout-joins

Given maps $i\colon A \to B$ and $j\colon K \to L$ of simplicial sets, define the *pushout-join* $i \circledast j$ as the map

$$i \circledast j\colon A * L \coprod_{A*K} B * K \xrightarrow{(i*\mathrm{id}_L, \mathrm{id}_B * j)} B * L. \tag{A.1}$$

If i and j are instead maps of marked simplicial sets, then $i \circledast j$ also defines a map of marked simplicial sets.

Lemma A.1.8. *Let S and T be two classes of morphisms, either both in* sSet *or in* sSet^+*. Then $\overline{S} \circledast \overline{T} \subseteq \overline{S \circledast T}$.*

Proof. For the case of sSet, this is [Rez22, Proposition 30.12]. The case of marked simplicial sets is a straightforward adaption of the argument thereof. □

Lemma A.1.9. *We have the following inclusions of classes of morphisms in* sSet *and* sSet^+*:*

$$\overline{\mathrm{RHorn}} \circledast \overline{\mathrm{Cell}} \subseteq \overline{\mathrm{InnHorn}} \quad \textit{and} \quad \overline{\mathrm{Cell}} \circledast \overline{\mathrm{LHorn}} \subseteq \overline{\mathrm{InnHorn}},$$

$$\overline{\mathrm{Cell}^\flat} \circledast \overline{\mathrm{RHorn}^\sharp} \subseteq \overline{\mathrm{RHorn}^\sharp} \quad \textit{and} \quad \overline{\mathrm{LHorn}^\sharp} \circledast \overline{\mathrm{Cell}^\flat} \subseteq \overline{\mathrm{LHorn}^\sharp}.$$

Proof. The results follow from lemma A.1.8 together with the following computation: for $j, k, n \in \mathbb{N}$ with $0 \leqslant j \leqslant n$, there are canonical isomorphisms

$$\begin{aligned} (\Lambda_j^n \subset \Delta^n) \circledast (\partial\Delta^k \subset \Delta^k) &\cong (\Lambda_j^{n+1+k} \subset \Delta^{n+1+k}), \\ (\partial\Delta^k \subset \Delta^k) \circledast (\Lambda_j^n \subset \Delta^n) &\cong (\Lambda_{k+1+j}^{n+1+k} \subset \Delta^n). \end{aligned}$$

□

The following computation will be essential in chapter 4.

Lemma A.1.10. *Let I and J be finite linear orders and $J_0 \subset J$ a suborder. Let i and j denote respectively the inclusions $\emptyset \subseteq I$ and $J_0 \subset J$. Then*

(1) the inclusion

$$\mathrm{id}_I * j\colon (\Delta^I)^\flat * (\Delta^{J_0})^\sharp \to (\Delta^I)^\flat * (\Delta^J)^\sharp$$

(2) and the inclusion

$$i \circledast j\colon (\Delta^J)^\sharp \underset{(\Delta^{J_0})^\sharp}{\cup} ((\Delta^I)^\flat * (\Delta^{J_0})^\sharp) \to (\Delta^I)^\flat * (\Delta^J)^\sharp$$

are both marked anodyne.

Proof. For both assertions, it suffices to show the result for $J = J_0 \cup \{y\}$.

(1) We consider the inclusion $\mathrm{id}_I * j$. We proceed by induction on the cardinality of I. Suppose first that $I = \emptyset$. Assuming that y is not an extremum in J, let y_- (respectively y_+) denote the maximum (resp. minimum) of the elements $x \in J$ such that $x < y$ (resp. $x > y$). Consider the spine inclusion $\mathrm{Sp}^J \to \Delta^J$, which is inner anodyne. Note that this map factors through the simplicial set $T = \Delta^{J_0} \cup \Delta^{y_- y y_+}$ as a inner anodyne inclusion $\mathrm{Sp}^J \to T$. As $\Delta^{J_0} \to \Delta^J$ also factors through T, it is enough to show that $(\Delta^J_0)^\sharp \to T^\sharp$ is marked anodyne: this follows from the two inclusions $(\Delta^{y_- y_+})^\sharp \to (\Lambda^{y_- y y_+}_{y_+})^\sharp \to (\Delta^{y_- y y_+})^\sharp$ being marked anodyne. The case where y is the maximum (resp. minimum) of J is analogous, replacing T by $\Delta^{J_0} \cup \Delta^{y_- y}$ (resp. $\Delta^{J_0} \cup \Delta^{y y_+}$).

For $I \neq \emptyset$, assume the result for finite linear orders of cardinality less than I. Let x be the minimum of I and let $I_0 = I \setminus \{x\}$. It suffices to show that the inclusion

$$(\Delta^{I_0} \subset \Delta^I) \circledast (\Delta^{J_0} \subset \Delta^J)\colon (\Delta^{I_0} * \Delta^J) \underset{\Delta^{I_0} * \Delta^{J_0}}{\cup} (\Delta^I * \Delta^{J_0}) \to (\Delta^I * \Delta^J)$$

is inner anodyne. This follows from lemma A.1.9, since $\Delta^{I_0} \subset \Delta^I$ is right anodyne and $\Delta^{I_0} \subset \Delta^I$ is a monomorphism.

(2) We now turn to the inclusion $i \circledast j$. If $y > J_0$, then j is marked left anodyne; using lemma A.1.9 we obtain that $i \circledast j$ is inner anodyne, hence a marked equivalence. Otherwise, we can partition J_0 as $J_0^- \coprod J_0^+$ such that $J_0^- < y < J_0^+$ and J_0^+ is non-empty. Then j factors as the composite

$$\Delta^{J_0} \longrightarrow \Delta^{J_0} \coprod_{\Delta^{J_0^+}} \left(\Delta^y * \Delta^{J_0^+}\right) \longrightarrow \Delta^J$$

where the first map is induced by the inclusion $e\colon \Delta^{J_0^+} \to \Delta^y * \Delta^{J_0^+}$ and the second map is $(\emptyset \subseteq \Delta^{J_0^-}) \circledast e$. Since e is marked right anodyne, using lemma A.1.9, we deduce that so is j and therefore also $i \circledast j$, as desired.

□

A.2 A note on cartesian fibrations and spaces of lifts

In this section, we recall a useful characterization of cartesianity of a functor in terms of contractibility of a certain space of lifts. Let $p\colon X \to S$ be an inner fibration of ∞-categories.

Definition A.2.1 (p-cartesian edges)**.** A morphism $f\colon x \to y$ in X is said to be *cartesian* if the canonical map

$$q_f\colon X_{/f} \longrightarrow X_{/y} \times_{S_{/py}} S_{/pf}$$

is a trivial fibration.

Notation A.2.2. Given a morphism $f\colon x \to y$ and an object z in X, base-changing q_f along $z\colon * \to X$ yields a functor

$$q_z\colon X_{/f} \times_X \{z\} \longrightarrow \mathcal{D}_z,$$

where $\mathcal{D}_z$ denotes the ∞-category $\left(X_{/y} \times_{S_{/py}} S_{/pf}\right) \times_X \{z\}$. The fiber of q_z at an object u will be denoted $\mathcal{L}$ and refered to as the *space of lifts* of u along f, leaving the dependance on (f, z, u) implicit in the notation. The situation is summarized in the following commutative diagram of ∞-categories

$$\begin{tikzcd}
\mathcal{L} \arrow[r] \arrow[d, "q_u"'] \arrow[dr, phantom, "\lrcorner", very near start] & X_{/f} \times_X \{z\} \arrow[r] \arrow[d, "q_z"] \arrow[dr, phantom, "\lrcorner", very near start] & X_{/f} \arrow[d, "q_f"] \\
* \arrow[r, "u"] & \mathcal{D}_z \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X_{/y} \times_{S_{/py}} S_{/pf} \arrow[d] \\
 & * \arrow[r, "z"] & X.
\end{tikzcd}$$

in which all squares are cartesian.

We will use the following equivalent description of cartesian edges, which is essentially a rewording of Proposition 2.4.4.3 in [Lur09a] and its proof.

Lemma A.2.3. *Let $f\colon x \to y$ be a morphism in X. Then f is p-cartesian if and only if for all $z \in X$, every fiber $\mathcal{L}$ of q_z is contractible.*

Proof. By Proposition 2.1.2.1 in [Lur09a], the morphism q_f is a right fibration, hence so are q_u and q_z. Now note that every fiber of q_f is of the form $\mathcal{L}$ for some choice of objects z and u. Since a right fibration is trivial if and only if each of its fibers is contractible, we get the result. □

Remark A.2.4. The proof also shows that $\mathcal{L}$ is a Kan complex, since q_u is a right fibration whose codomain is a Kan complex. This justifies the use of the terminology *space* of lifts for $\mathcal{L}$.

Definition A.2.5 (Cartesian fibrations)**.** The functor $p\colon X \to S$ is a *cartesian fibration* if for all $y \in X$, every morphism $\overline{x} \to p(y)$ in S admits a lift $x \to y$ along p which is p-cartesian.

A.3 Principal ∞-bundles

In this section, we recall some definitions and basic properties of groupoid objects, ∞-groups and principal bundles in higher category theory. We mostly follow the exposition of [NSS15].

Let $\mathcal{T}$ be an ∞-topos.

Definition A.3.1 ([Lur09a, Definition 6.1.2.7])**.** A *groupoid object* in $\mathcal{T}$ is a simplicial object $G_\bullet \colon \Delta^{\mathrm{op}} \to \mathcal{T}$ such that for every $n \in \mathbb{N}$ and every partition $[k] \cup [k'] = [n]$ with $[k] \cap [k'] = \{*\}$, the induced diagram

$$\begin{tikzcd} G_n \arrow[r] \arrow[d] & G_k \arrow[d] \\ G_{k'} \arrow[r] & G_0 \end{tikzcd} \tag{A.2}$$

is a pullback in $\mathcal{T}$. The full subcategory of $\mathrm{Fun}(\Delta^{\mathrm{op}}, \mathcal{T})$ on the groupoid objects is denoted $\mathrm{Grpd}(\mathcal{T})$.

Definition A.3.2. For $f \colon X \to Y$ a morphism in $\mathcal{T}$, one has a associated groupoid object $\check{C}(X \to Y)$ in $\mathcal{T}$ called the Čech nerve of f given in degree n by the $(n+1)$-fold fiber product

$$\check{C}(X \to Y)_n = X \times_Y X \times_Y \cdots \times_Y X.$$

We say that f is an *effective epimorphism* if it is the colimiting cocone of its Čech nerve, i.e. if we may write

$$f \colon X \longrightarrow \check{C}(f).$$

Let $\mathrm{Eff}(\mathcal{T})$ denote the full subcategory of $\mathrm{Fun}(\Delta^1, \mathcal{T})$ on the effective epimorphisms.

Proposition A.3.3 ([Lur09a, Corollary 6.2.3.5])**.** *The Čech nerve construction provides an equivalence of ∞-categories*

$$\check{C} \colon \mathrm{Eff}(\mathcal{T}) \simeq \mathrm{Grpd}(\mathcal{T}) \tag{A.3}$$

whose inverse sends a groupoid $G_\bullet$ to the colimiting cocone $G_0 \to \operatorname{colim} G_\bullet$.

Definition A.3.4. An ∞-*group in* $\mathcal{T}$ is a groupoid $G_\bullet$ in $\mathcal{T}$ such that $G_0 \simeq *$. The corresponding full subcategory of $\mathrm{Grpd}(\mathcal{T})$ is denoted $\mathrm{Grp}(\mathcal{T})$. We usually write G for the space G_1 and will often abuse notation by refering to G as the ∞-group, leaving the rest of the simplicial structure $G_\bullet$ implicit.

We now recall the delooping equivalence.

Proposition A.3.5 ([Lur09a, Lemma 7.2.2.11])**.** *The loop space functor Ω canonically extends to a functor from the ∞-category $\mathcal{T}_*$ of pointed objects in $\mathcal{T}$ to* $\mathrm{Grp}(\mathcal{T})$. *Its restriction to connected pointed objects yields an equivalence of ∞-categories*

$$\Omega \colon (\mathcal{T}_*)_{\geqslant 1} \simeq \mathrm{Grp}(\mathcal{T}) \colon \mathrm{B}.$$

The functor B inverse to Ω is called the *delooping*, or *classifying space* functor. The effective epimorphism $* \to \mathrm{B}G$ associated to an ∞-group G is the colimiting cocone

$$\begin{tikzcd} \dots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \arrow[r, shift left=1] \arrow[r, shift right=1] & * \arrow[r] & \mathrm{B}G \end{tikzcd}$$

induced by the simplicial object $G\colon \Delta^{\mathrm{op}} \to \mathfrak{T}$.

Definition A.3.6 ([NSS15, Definition 3.1])**.** Let $G_\bullet \in \mathrm{Grp}(\mathfrak{T})$ be a group object and X an object in $\mathfrak{T}$. A *G-action on X* is a groupoid object $(X//G)_\bullet$ in $\mathfrak{T}$ of the form

$$\begin{tikzcd} \dots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & X \times G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & X \times G \arrow[r, shift left=1] \arrow[r, shift right=1, "\mathrm{proj}"'] & X \end{tikzcd}$$

such that the degreewise projection maps $X \times G^n \to G^n$ yield a morphism of groupoid objects $(X//G)_\bullet \to G_\bullet$. The ∞-*quotient* of the action is the colimit object $X//G := \mathrm{colim}(X//G)$ in $\mathfrak{T}$.
The ∞-category $G\mathrm{Action}(\mathfrak{T})$ of G-actions in $\mathfrak{T}$ is the full subcategory of $\mathrm{Grpd}(\mathfrak{T})_{/G_\bullet}$ on G-actions.

Definition A.3.7 ([NSS15, Definition 3.4])**.** Let $G_\bullet \in \mathrm{Grp}(\mathfrak{T})$ be a group object and X an object in $\mathfrak{T}$. A *G-principal ∞-bundle* over X is a morphism $Y \to X$ in $\mathfrak{T}$ together with a G-action on Y, such that $Y \to X$ exhibits X as the quotient $Y//G$.
The ∞-category $G\mathrm{Bun}(X)$ of G-principal ∞-bundles over X is the homotopy fiber at X of the quotient functor

$$G\mathrm{Action}(\mathfrak{T}) \subseteq \mathrm{Grpd}(\mathfrak{T})_{/G_\bullet} \longrightarrow \mathrm{Grpd}(\mathfrak{T}) \overset{\mathrm{colim}}{\longrightarrow} \mathfrak{T}.$$

The following result will be useful in chapter 5.

Proposition A.3.8 ([NSS15, Proposition 3.8])**.** *If G is an ∞-group and $X \to \mathrm{B}G$ a morphism in $\mathfrak{T}$, then its homotopy fiber $Y \to X$ at the distinguished point of $\mathrm{B}G$ carries a canonical structure of a G-principal ∞-bundle over X.*

The G-principal ∞-bundle structure is obtained by considering the following morphism of augmented simplicial objects

$$\begin{tikzcd} \dots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & Y \times G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] \arrow[d] & Y \times G \arrow[r, shift left=1] \arrow[r, shift right=1, "\mathrm{proj}"'] \arrow[d] & Y \arrow[r] \arrow[d] & X \arrow[d] \\ \dots \arrow[r, shift left=3] \arrow[r, shift left=1] \arrow[r, shift right=1] \arrow[r, shift right=3] & G \times G \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & G \arrow[r, shift left=1] \arrow[r, shift right=1] & * \arrow[r] & \mathrm{B}G \end{tikzcd}$$

in which all rectangle are cartesian squares.

Moreover, all principal ∞-bundles are obtained through this construction, as stated in the next result.

Theorem A.3.9 (Classification of G-principal ∞-bundles, [NSS15, Theorem 3.17])**.** *For all ∞-groups $G \in \mathrm{Grp}(\mathfrak{T})$ and all objects $X \in \mathfrak{T}$, there is a natural equivalence of ∞-groupoids*

$$G\mathrm{Bun}(X) \simeq \mathrm{Map}_{\mathfrak{T}}(X, \mathrm{B}G)$$

given on objects by the construction $(p\colon X \to \mathrm{B}G) \mapsto (\mathrm{hofib}(p) \to X)$ *of proposition A.3.8.*

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